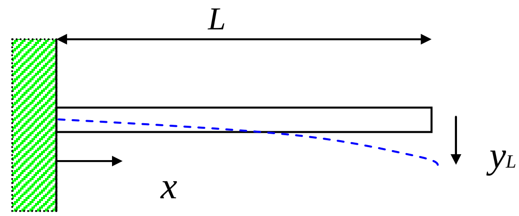
Appendix D. Technical Note on Assumed Modes

The **fundamental vibrating mode** of a cantilever beam and its associated natural frequency can be modeled as a single degree of freedom **lumped mass on a spring**.



The beam equivalent stiffness and mass can be determined by equating the beam strain energy *(V)* and kinetic energy *(T)* of the vibrating beam to the strain and kinetic energy of the lumped spring and mass, respectively. The equivalent displacement coordinate should be equal for both energies.

The beam has continuously distributed mass and elastic properties. Let

ρ : beam material density, *E:* beam elastic modulus,
A : beam cross section area, *L:* beam length,
I: area moment of inertia.

$y_{(x,t)}$ is the displacement of a beam material point, i.e. a function of its location *(x)* and time *(t)*. $y_{L(t)}$ denotes the beam dynamic displacement at $x=L$. The beam potential (strain) and kinetic energies, V and T, are defined as:

$$V = \frac{1}{2}\int_0^L E\,I\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx = \frac{1}{2}K_{eq}\,y_L^2;$$

(1)

$$T = \frac{1}{2}\int_0^L \rho\,A\left(\frac{\partial^2 y}{\partial t^2}\right)^2 dx = \frac{1}{2}M_{eq}\,\dot{y}_L^2$$

In practice, an assumed shape of vibration $\phi_{(x)}$ is used to estimate the equivalent stiffness (K_{eq}) and mass (M_{eq}).

Let $$y_{(x,t)} = \phi_{(x)}\, y_{L(t)} \tag{2}$$

The **mode shape** $\phi_{(x)}$ must be twice differentiable and consistent with the essential boundary conditions of the **cantilever beam**, i.e. no displacement or slope at the fixed end. That is, from

$y_{(0,t)} = 0 \quad \rightarrow \quad \phi_{(x=0)} = 0$

$(\partial y/\partial x)_{x=0} = 0 \quad \rightarrow \quad d\phi/dx|_{x=0} = 0$ for all times $t > 0$.

Substitution of $y_{(x,t)} = \phi_{(x)}\, y_{L(t)}$ into Eq. (1) gives:

$$K_{eq} = \int_0^L E\,I \left(\frac{d^2 \phi}{d\,x^2} \right)^2 dx; \qquad M_{eq} = \int_0^L \rho\, A (\phi)^2\, dx \tag{3}$$

The fundamental natural frequency of the vibrating beam is then given by

$$\omega_n = \sqrt{K_{eq} / M_{eq}} \tag{4}$$

Using $\phi = (x/L)^2$, then $M_{eq} = \frac{1}{5} \rho\, A\, L$; $K_{eq} = 4 \frac{E\,I}{L^3}$;

And $$\omega_n \approx \frac{1}{L^2} \left(20 \frac{E\,I}{\rho\, A} \right)^{1/2} \tag{5}$$

The equivalent mass of the beam, M_{eq}, is a fraction of the total mass (~1/5) since the material points composing the beam participate differently in the vibratory motion.

$K_{eq}=3EI/L^3$ (more exact value) follows if the **static deflection curve** for the beam with a point load at its free end is used as the assumed mode shape, i.e.

$$\phi(x)=\frac{1}{2}\left[3\left(\frac{x}{L}\right)^2-\left(\frac{x}{L}\right)^3\right]$$

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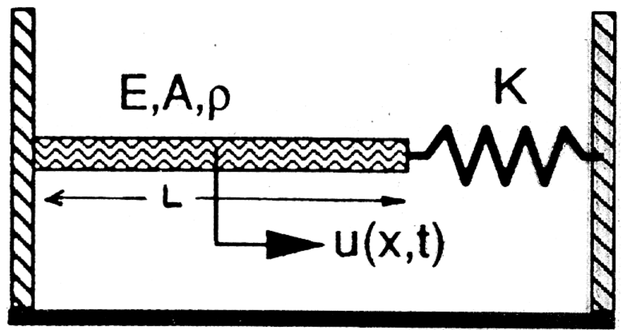
Find Natural frequency



SOLUTION: Find Approximate natural frequency Use Assumed Modes method.

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 dx \quad ; \quad V = \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{1}{2}K\,U^2_{(L,t)} \qquad (1)$$

Assume mode: let $u(x,t) = \psi(x)\,U(t)$ (2)

where $\psi(0) = \emptyset$ by essential B.C. (3)

$\psi(x) \in C^1(0,L)$

Then

$$T = \frac{1}{2}\left[\int_0^L \rho A \psi^2 dx\right]\dot{U}^2 = \frac{1}{2} M_{eq}\,\dot{U}^2$$

$$V = \frac{1}{2}\left[\int_0^L AE(\psi')^2 dx + K\psi^2_{(L)}\right]U^2 = \frac{1}{2} K_{eq}\,U^2 \qquad (4)$$

Equivalent mass and stiffness are

$$M_{eq} = \int_0^L \rho A\psi^2 dx, \quad K_{eq} = \int_0^L EA\left(\frac{d\psi}{dx}\right)^2 dx + K\psi^2_{(L)}, \text{ and } \omega_n = \sqrt{\frac{K_{eq}}{M_{eq}}}$$

Select $\psi(x) = x/L$ satisfying (3)

$$M_{eq} = \rho AL/3 \; ; \; K_{eq} = \frac{EA}{L} + K$$

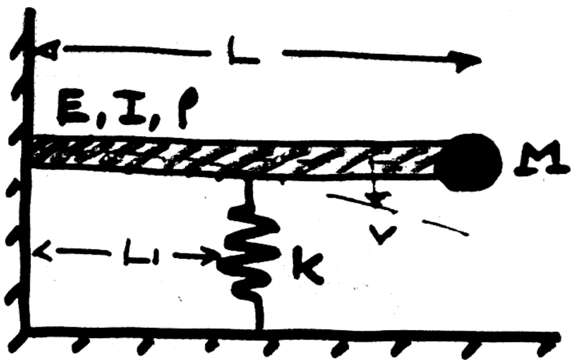
$$\omega_n = \left[\frac{EA/L + K}{\rho AL/3}\right]^{1/2}$$

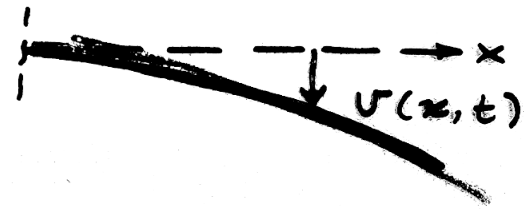
Rayleigh-Ritz Method.

Assumed Modes Method for CONTINUOUS SYSTEMS:

Multiple Degree of Freedom ANALYSIS

Consider the lateral vibrations of the following system:





The total kinetic energy and potential energy of the system are:

$$T = \frac{1}{2} M \dot{v}^2(L,t) + \frac{1}{2}\int_0^L \rho A \left(\frac{\partial v}{\partial t}\right)^2 dx \qquad (1)$$

$$V = \frac{1}{2} K\, v^2(L_1,t) + \frac{1}{2}\int_0^L EI \left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx \qquad (2)$$

Now, let's assume the displacement vector $v(x,t)$ is given by

$$v(x,t) = \sum_{i=1}^{n} \psi_i(x)\, V_i(t) \qquad (3)$$

where the set $\{\psi_i(x)\}_{i=1}^{n}$ is a set of Assumed Modes functions

$\boxed{\text{ADMISSIBLE}}$ to the problem, i.e.

$$\psi_i \in C^2(x \in [0,L]) \qquad (4)$$

ψ_i satisfy essential boundary conditions (for the example $\left.\begin{aligned}\psi(0)&=0\\ d\psi/dx(0)&=0\end{aligned}\right\}$ only!

AND : $\nu_i(t)$, $i=1,2,\dots n$ are discrete displacements associated with each mode shape.

Linear independence of the shape functions ψ is not required. (later explained)

Note that $\dot{v} = \dfrac{\partial v}{\partial t} = \sum_i^n \psi_i \, \dot{\nu}_i(t)$

$$\frac{\partial^2 v}{\partial x^2} = \sum_i^n \frac{d^2\psi_i}{dx^2} \cdot \nu_i(t) \qquad (5)$$

Substitution of (3) into (1) & (2) gives the following approximate T & V.

$$T = \frac{1}{2} M \sum_i^n \psi_i^2(L)\, \dot{\nu}_i^2 + \tag{6}$$

$$\frac{1}{2} \int_0^L \rho A \underline{\left(\sum_i^n \psi_i \dot{\nu}_i \right)\left(\sum_j \psi_j \dot{\nu}_j \right)} dx$$

$$V = \frac{1}{2} K \sum_i^n \psi_i^2(L)\, \nu_i^2 + \tag{7}$$

$$\frac{1}{2} \int_0^L EI \left(\sum_i^n \psi_i'' \nu_i \right)\left(\sum_j \psi_j'' \nu_j \right) dx$$

where $\psi_i'' = \dfrac{d^2 \psi_i}{dx^2}$.

now using: $\displaystyle \int_0^L \sum_i = \sum_i \int_0^L$

we get from (6) & (7)

$$T \cong \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ M \underbrace{\psi_i^2(L)}\, \delta_{ij} + \int_0^L \rho A \psi_i \psi_j \, dx \right\} \dot{\nu}_i \dot{\nu}_j \tag{8}$$

where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

and

$$V \cong \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ k \psi_i^2(L_1) \delta_{ij} + \int_0^L EI \psi_i'' \psi_j'' \, dx \right\} \nu_i \nu_j \tag{9}$$

Let's define:

$$\boxed{M_{ij} = M \psi_i^2(L)\, \delta_{ij} + \int_0^L \rho A \psi_i \psi_j \, dx} \tag{10}$$

$i, j = 1, 2, \dots n$

$$\boxed{K_{ij} = k \psi_i^2(L_1)\, \delta_{ij} + \int_0^L EI \psi_i'' \psi_j'' \, dx} \tag{11}$$

as the system mass and stiffness elements

<u>Note</u>: $M_{ij} = M_{ji}$

$K_{ij} = K_{ji}$ i.e $\Rightarrow$ symmetric positive-definite r d write (8) & (9) as:

$$T \simeq \frac{1}{2} \sum_{i}^{n} \sum_{j}^{n} M_{ij} \dot{v}_i \dot{v}_j \qquad (12)$$

$$V \simeq \frac{1}{2} \sum_{i} \sum_{j} K_{ij} v_i v_j \qquad (13)$$

or letting $\mathbb{M} = \begin{bmatrix} M_{11} & M_{12} & \cdots & \\ M_{21} & & & \\ & & & M_{nn} \end{bmatrix}$ $\quad \mathbb{M} = \mathbb{M}^T$

(14)

$$\mathbb{K} = \begin{bmatrix} K_{11} & K_{12} & & \\ K_{21} & & \ddots & \\ & & & K_{nn} \end{bmatrix} \qquad \mathbb{K} = \mathbb{K}^T$$

be the system mass and stiffness matrices

and

$$\mathbb{v}^T = \{ v_1 \ v_2 \ \cdots \ v_n \} \qquad (15)$$

we write (12) & (13) as:

$$\boxed{T \simeq \frac{1}{2} \dot{\mathbb{v}}^T \mathbb{M} \dot{\mathbb{v}}} \qquad (16)$$

$$\boxed{V \simeq \frac{1}{2} \mathbb{v}^T \mathbb{K} \mathbb{v}} \qquad (17)$$

<u>Now</u> we have n DOF's: $\nu_1, \nu_2, \ldots \nu_n$ so we can use Lagrange's equation of motion as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\nu}_q}\right) + \frac{\partial V}{\partial \nu_q} = Q_q = 0 \qquad (18)$$
$$q = 1, 2, \ldots n$$

to determine the eqn. of motion.

From (13) $V \simeq \frac{1}{2} \sum_i \sum_j K_{ij} \nu_i \nu_j$

$$\frac{\partial V}{\partial \nu_q} = \frac{1}{2} \sum_i \sum_j K_{ij}\left(\frac{\partial \nu_i}{\partial \nu_q} \nu_j + \frac{\partial \nu_j}{\partial \nu_q} \nu_i\right) \qquad (a)$$

since $K_{ij} \neq fn(\nu_q)$

and using: $\frac{\partial \nu_i}{\partial \nu_q} = \delta_{iq} = \begin{cases} 1 & \text{if } i = q \\ 0 & \text{if } i \neq q \end{cases} \qquad (b)$

we get:

$$\frac{\partial V}{\partial \nu_q} = \frac{1}{2} \sum_i \sum_j K_{ij}\left(\delta_{iq} \nu_j + \delta_{jq} \nu_i\right)$$

$$= \frac{1}{2} \sum_j K_{qj} \nu_j + \frac{1}{2} \sum_q K_{iq} \nu_i$$

but since $K_{qi} = K_{iq}$ (symmetric)

Then :

$$\frac{\partial V}{\partial \nu_q} = \frac{1}{2} \sum_j K_{qj} \nu_j + \frac{1}{2} \sum_{\cancel{j}\, i} K_{qi} \nu_i \qquad (c)$$

i & j are "dumys" here

so we express (c) as:

$$\frac{\partial V}{\partial \nu_q} = \frac{1}{2} \sum_j K_{qj} \nu_j + \frac{1}{2} \sum_{\cancel{i}\, j} K_{qj} \nu_j \qquad (d)$$

or $$\frac{\partial V}{\partial \nu_q} = \sum_j K_{qj} \nu_j = \mathbb{K}\, \boldsymbol{\nu} \qquad (19)$$

Similarly we can show that:

$$\frac{\partial T}{\partial \dot{\nu}_q} = \sum_j M_{qj} \dot{\nu}_j = \mathbb{M}\, \dot{\boldsymbol{\nu}} \qquad (20)$$

and $$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\nu}_q}\right) = \sum_j M_{qj} \ddot{\nu}_j = \mathbb{M}\, \ddot{\boldsymbol{\nu}}} \qquad (21)$$

here we have used $\mathbb{M}^T = \mathbb{M}$

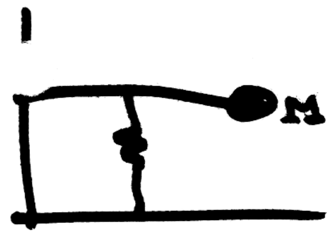
Thus, the equation of motion are equal to:

$$\boxed{\mathbb{M}\,\ddot{\boldsymbol{\nu}} + \mathbb{K}\,\boldsymbol{\nu} = \mathbf{0}} \qquad (22)$$

or

$$\sum_i^n M_{ij}\,\ddot{\nu}_j + \sum_i^n K_{ij}\,\nu_j = 0_i \qquad (23)$$

Now, for the problem of interest, the elements of the mass & stiffness matrices are:



$$M_{ij} = \int_0^L \rho A \phi_i \phi_j \, dx + M\, \phi_i(L)\,\phi_j(L)\,\delta_{ij} = M_{ji}$$

$$K_{ij} = \int_0^L EI\, \phi_i'' \phi_j'' \, dx + K\, \phi_i(L)\,\phi_j(L)\,\delta_{ij} = K_{ji}$$

$i, j = 1, 2, \ldots n$

EXERCISE: what are the eqns. of motion if $\mathbb{M}$ and $\mathbb{K}$ are not symmetric?

Summary:

Using the assumed modes method, a continuous system can be modeled as n-DOF discrete system with equations $\mathbb{M}\ddot{v} + \mathbb{K}v = Q$

This was the way done in the past to handle continuous systems of complex shapes.

Disadvantages:

1) difficult to find more than one shape function (assumed mode)
2) $\{\psi_i\}_{i=1}^{n}$ need to be defined over entire domain, and
3) $\mathbb{M}$ and $\mathbb{K}$ matrices are usually full of elements.

If $\{\psi_i\}_{i=1}^{n}$ are linearly independent the problem is very much reduced since in this case the mass and stiffness matrices are diagonal!

Here, we mean linear independence of the shape functions in the following way:

$$\int_0^L \rho A \, \psi_i \psi_j \, dx = 0 \quad \text{if} \quad i \neq j$$

$$\int_0^L EI \, \psi_i'' \psi_j'' \, dx = 0 \quad \text{if} \quad i \neq j \tag{24}$$

i.e. the functional inner products are zero!

This is very difficult to achieve! for most cases.

$$\begin{bmatrix} \ddots & & \\ & M & \\ & & \ddots \end{bmatrix} ; \quad \begin{bmatrix} \ddots & & \\ & K & \\ & & \ddots \end{bmatrix}$$