2DOF undamped system - MODAL ANALYSIS

ORIGIN := 1

The BEATING phenomenon

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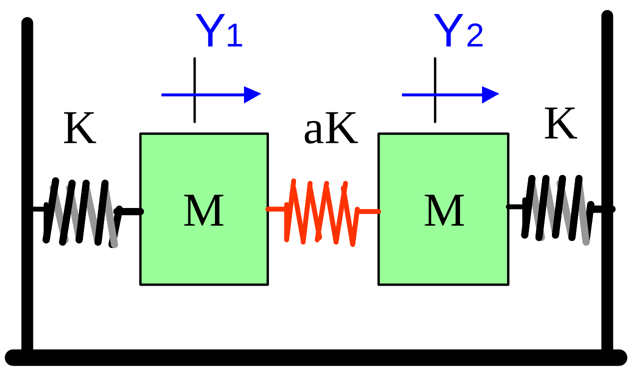
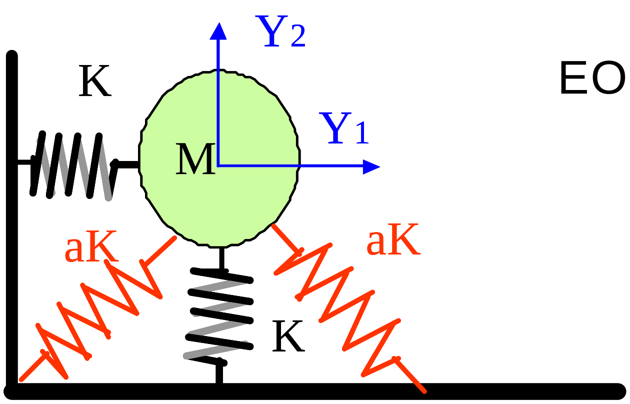
2 DOF system with weak coupling



EOMs:

$$M\ddot{Y}_1 = -K\,Y_1 + aK\left(Y_2 - Y_1\right)$$
$$M\ddot{Y}_2 = -aK\left(Y_2 - Y_1\right) - K\,Y_2$$

2 DOF system with weak cross-coupling



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Schematic only

$$M\begin{bmatrix}\ddot{Y}_1\\ \ddot{Y}_2\end{bmatrix} + \begin{bmatrix}K(1+a) & -aK\\ -aK & K(1+a)\end{bmatrix}\begin{bmatrix}Y_1\\ Y_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Given EOMs for a 2DOF - undamped- system - WEAK structural coupling:

$$\begin{pmatrix} m_o & 0 \\ 0 & m_o \end{pmatrix} \cdot \frac{d^2}{dt^2}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{bmatrix} k_o\cdot(1+a) & -k_o\cdot a \\ -k_o\cdot a & k_o\cdot(1+a) \end{bmatrix} \cdot \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad a<1 \qquad (1)$$

(a) FIND NATURAL FREQUENCIES AND NATURAL MODES:
Assume the motions are periodic with frequency ω, ie

$$Y_1 = a_1\cdot\cos(\omega\cdot t) \qquad Y_2 = a_2\cdot\cos(\omega\cdot t) \qquad (2)$$

Let $\lambda = \omega^2$

Substitution of (2) into (1) gives the homogeneous system of eqns

$$\begin{bmatrix} k_o\cdot(1+a) - m_o\cdot\lambda & -k_o\cdot a \\ -k_o\cdot a & k_o\cdot(1+a) - m_o\cdot\lambda \end{bmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (3)$$

has a non-trivial solution if the determinant of the system of equations equals zero, i.e. if

$$\Delta(\omega) = \left[k_o\cdot(1+a) - m_o\cdot\lambda\right]^2 - k_o^{\,2}\cdot a^2 = 0$$

Expanding the products in the determinant gives the **system characteristic equation**

$$0 = \lambda^2 m_o^{\,2} - \lambda\cdot m_o\cdot 2\cdot k_o\cdot(1+a) + k_o^{\,2}\cdot(1+2\cdot a)$$

with $\bar{\lambda} = \lambda\cdot\left(\frac{m_o}{k_o}\right)$ Leads to: $0 = \left[1\cdot\left(\bar{\lambda}\right)^2 + b\cdot\bar{\lambda} + c\right]$ (4)

with: $a := .1$

$b := -2\cdot(1+a)$

$c := (1+2\cdot a)$

The roots (eigenvalues) of the characteristic equation are

$$\lambda_1 := \frac{-b - \left(b^2 - 4\cdot c\right)^{0.5}}{2} \qquad \lambda_2 := \frac{-b + \left(b^2 - 4\cdot c\right)^{0.5}}{2}$$

$$b^2 - 4\cdot c = -2 - 4\cdot a + 2\cdot a^2$$

$$\lambda = \begin{pmatrix} 1 \\ 1.2 \end{pmatrix} \left(\frac{k_o}{m_o}\right)$$

and the **natural frequencies** are:

$$\omega_1 := \left(\lambda_1\right)^{0.5} \qquad \omega_2 := \left(\lambda_2\right)^{0.5} \qquad \omega = \begin{pmatrix} 1 \\ 1.095 \end{pmatrix} \left(\frac{k_o}{m_o}\right)^{0.5}$$

Find the eigenvectors:

The two equations in Eq. (3) are **linearly dependent**. Thus, one cannot solve for a1 and a2.

for ω_i $\qquad \phi_1 = 1 \qquad \phi_2 = \frac{\left[k_o\cdot(1+a) - \lambda_i\cdot k_o\right]}{k_o\cdot a}$

$$\phi_1 := \left(1 \quad \frac{1 + a - \lambda_1}{a}\right)^T \qquad \phi_2 := \left(1 \quad \frac{1 + a - \lambda_2}{a}\right)^T$$

$$\phi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \phi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are the eigenvectors (mode shapes)

(c) find the numerical value for each natural frequency:

$$m_o := \frac{2000\text{lbf}}{g} \qquad k_o := 10^5 \cdot \frac{\text{lbf}}{\text{in}}$$

Since $\omega := \omega \cdot \left(\frac{k_o}{m_o}\right)^{0.5}$

$$\omega = \begin{pmatrix} 138.94 \\ 152.202 \end{pmatrix} \frac{\text{rad}}{\text{sec}}$$

Note that mass must be expressed in physical units consistent with the problem, i.e.

$$f_n := \frac{\omega}{2 \cdot \pi} \qquad f_n = \begin{pmatrix} 22.113 \\ 24.224 \end{pmatrix} \text{Hz}$$

$$m_o = 166.667 \frac{\text{ft}}{\text{sec}^2} \frac{\text{lb} \cdot \text{sec}^2}{\text{in}}$$

automated calculation

(b)MODAL ANALYSIS

(b.1) FIND modal masses and stiffnesses

$$M_M := \Phi^T \cdot M \cdot \Phi \qquad K_M := \Phi^T \cdot K \cdot \Phi$$

$$M_M = \begin{pmatrix} 10.36 & 0 \\ 0 & 10.36 \end{pmatrix} \frac{\text{lbf} \cdot \text{sec}^2}{\text{in}}$$

non-diagonal elements are very small= non zero b/c of roundoff in numerical calculator

modal masses and stiffnesses:

Mode 1 $\quad M_{m_1} := M_{M_{1,1}} \qquad K_{m_1} := \left(\omega_{n_1}\right)^2 \cdot M_{M_{1,1}}$

Mode 2 $\quad M_{m_2} := M_{M_{2,2}} \qquad K_{m_2} := \left(\omega_{n_2}\right)^2 \cdot M_{M_{2,2}}$

$$M_m = \begin{pmatrix} 10.36 \\ 10.36 \end{pmatrix} \frac{\text{lbf} \cdot \text{sec}^2}{\text{in}}$$

$$K_m = \begin{pmatrix} 2 \times 10^5 \\ 2.4 \times 10^5 \end{pmatrix} \frac{\text{lbf}}{\text{in}}$$

Let $\Phi_{inv} := \Phi^{-1}$

FREE RESPONSE ONLY

At time t=0s, find initial conditions in modal coordinates $Y_o := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot in$ $V_o := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \frac{in}{sec}$

$q_o := \Phi_{inv} \cdot Y_o$ $q_{o\_dot} := \Phi_{inv} \cdot V_o$ $F := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot lbf$

:

$q_o = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} in$ $q_{o\_dot} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \frac{in}{sec}$ Both natural modes will be excited.

Calculate the modal force

$Q := \Phi^T \cdot F$ $Q = \begin{pmatrix} 0 \\ 0 \end{pmatrix} lbf$

(c.3) Modal EOMs and modal responses $M_{m_i} \left(\frac{d^2}{dt^2} q_i \right) + K_{m_i} \cdot q_i = 0$ $i = 1, 2$

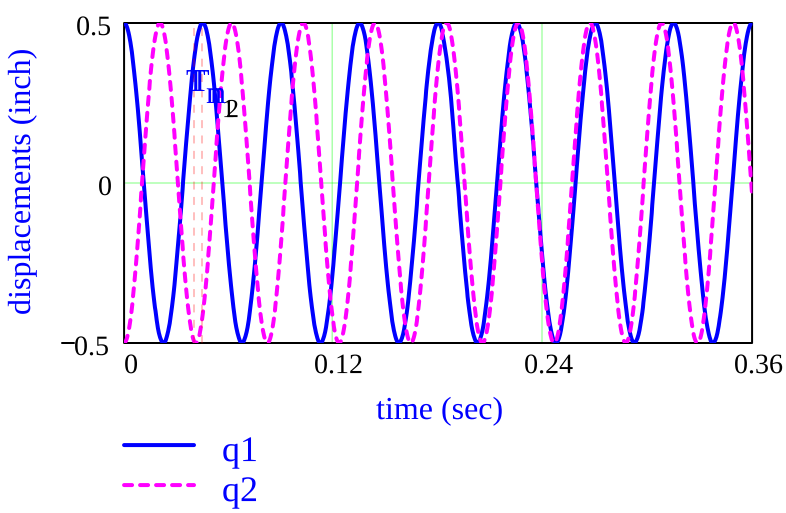
Using the cheat sheet, and since the Initial conditions are null, the response in modal coordinates are

$$q_1(t) := q_{o_1} \cdot \cos\left(\omega_{n_1} \cdot t\right) + \frac{q_{o\_dot_1}}{\omega_{n_1}} \cdot \sin\left(\omega_{n_1} \cdot t\right)$$

$$q_2(t) := q_{o_2} \cdot \cos\left(\omega_{n_2} \cdot t\right) + \frac{q_{o\_dot_2}}{\omega_{n_2}} \cdot \sin\left(\omega_{n_2} \cdot t\right)$$

$\omega_n = \begin{pmatrix} 138.94 \\ 152.202 \end{pmatrix} \frac{rad}{sec}$

<u>GRAPH</u>: Modal responses q1 and q2 vs. time.



fundamental periods

$T_n = \begin{pmatrix} 0.045 \\ 0.041 \end{pmatrix} sec$

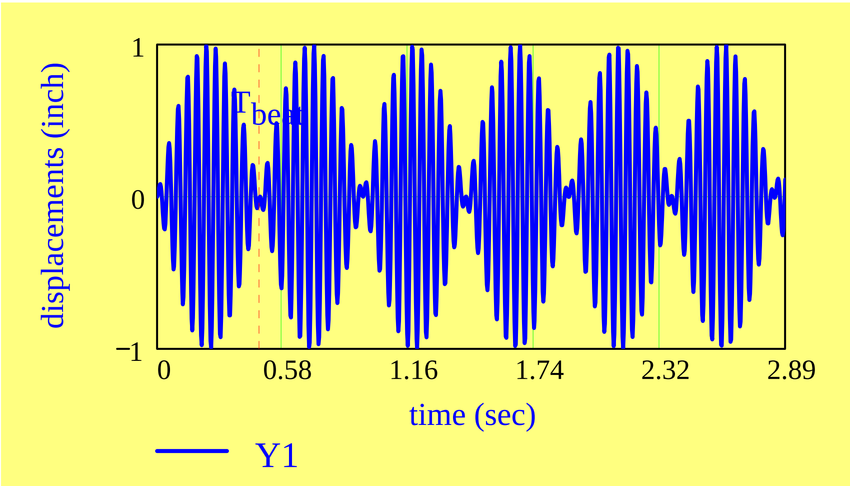
(b.4) The response in physical coordinates, equals $Y = \Phi q$

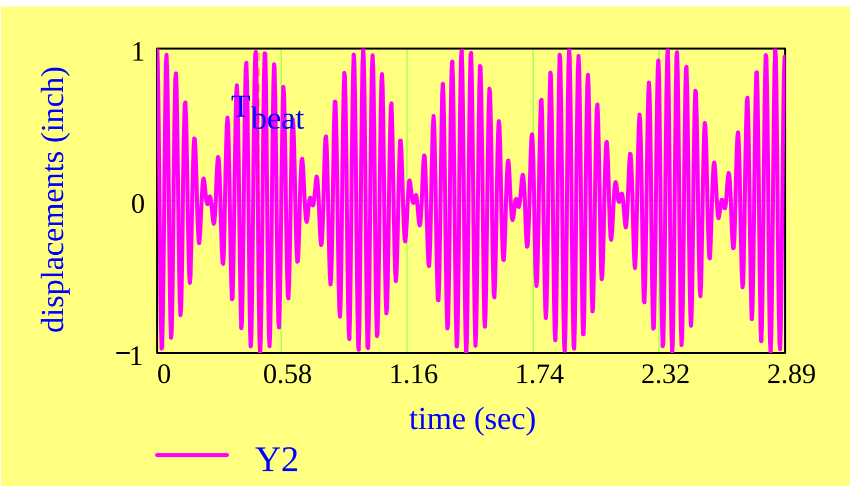
$$Y = \Phi q$$

$$Y_1(t) := q_1(t) + q_2(t)$$

with $\Phi = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$Y_2(t) := \Phi_{2,1} \cdot q_1(t) + \Phi_{2,2} \cdot q_2(t)$$





Note that there is no damping or attenuation of motions.

fundamental periods $T_n = \begin{pmatrix} 0.045 \\ 0.041 \end{pmatrix} \text{sec}$

$f_{ave} = 23.12\,\text{Hz}$ $f_{beat} = 2.111\,\text{Hz}$

$T_{beat} = 0.474\,\text{sec}$

$\dfrac{T_{beat}}{T_{ave}} = 10.954$ $a = 0.1$

$$\omega_{beat} := (\omega_2 - \omega_1)$$

$$T_{beat} := \frac{2 \cdot \pi}{\omega_{beat}}$$

$$f_n = \begin{pmatrix} 22.113 \\ 24.224 \end{pmatrix} \text{Hz}$$

Note that if a~0 (small value), the beating phenomenon appears. There are two distinct frequencies:

$f_{ave} := \dfrac{\left(f_{n_1} + f_{n_2}\right)}{2}$ $\Delta\omega := (\omega_2 - \omega_1)$ $T_{envelope} := \dfrac{2 \cdot \pi}{\Delta\omega}$

$f_{ave} = 23.168\,\text{Hz}$

$T_{envelope} = 0.474\,\text{sec}$ period of "low frequency" envelope

$\Delta\omega = 13.261 \dfrac{\text{rad}}{\text{sec}}$

Calculate FFT

FFT amplitude for signal 1

$max_f := 50 \cdot Hz$ plot

$\Delta f = 0.346\,Hz$