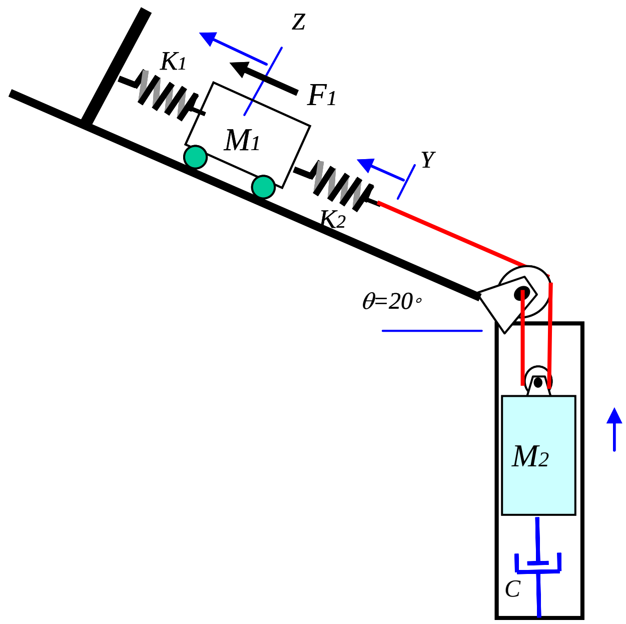
EXAMPLE PROBLEM for MEEN 363 – SPRING 06

Objectives:

a) To derive EOMS of a 2DOF system
b) To understand concept of static equilibrium
c) To learn the correct usage of physical units (US system)
d) To calculate natural frequencies and natural mode shapes
e) To predict the response of a system using modal coordinates, case: constant amplitude load
f) To determine the final position of the system once transient effects have disappeared
g) To learn how to combine mathematical statements with explanatory sentences.

Car 1 must pull a heavy block stuck in a hollow and deep mining shaft. The front end of car 1 is tied to a big tree with a cable of stiffness K_1. A flexible cable of stiffness K_2 is connected to an inextensible cable that in turn, with a pulley system, is connected to the block[1]. The motorcar engine will drive the car upward with force $F_1(t)$ known. The damping coefficient *(C)* represents the viscous drag between the block and shaft walls. In the figure, ***Y=X=Z=0*** denote the **static equilibrium position** (SEP) of the system.



SEP means no motion of car and block, and engine turned off, F_1=0. Thus, at the SEP springs K_1 and K_2 are already deflected. For example, spring 2 must support 50% of the block weight (W_2) as easily seen from the cable & pulley constraint. Next, spring 1 must also develop a static force to hold 50% of W_2 plus a fraction of the car 1 weight, i.e. $W_1 \sin(20°)$. This knowledge is BASIC, does not require of elaborate thinking or deriving lengthy equations.

a) Identify the kinematical constraint relating motions *Y* and *X*. The cable does NOT slip on the pulley.
b) Draw free body diagrams for the car and block, label all forces and show their constitutive relation in terms of the motion coordinates, if applicable.
c) Determine the <u>**static**</u> deflection (δ_s) of each spring element
d) Derive EOMs for the car and block motion in terms of coordinates *Z & X*.

For items (c) & (e-f-g) use

$K_2 := 10^5 \cdot \frac{lb}{in}$ $\quad W_2 := 5000\,lb$ $\quad C := 1500\,lb \cdot \frac{sec}{in}$

$K_1 := 10^5 \cdot \frac{lb}{in}$ $\quad W_1 := 1000\,lb$ $\quad \theta := 20 \cdot \frac{\pi}{180}$

e) Find the system (undamped) natural frequencies and mode shapes, i.e. solve for the system eigenvalues and eigenvectors.
f) For $F_1(t)$=1000 lbf find the (undamped) response of the system using modal analysis
g) Find the final or terminal position of the system, i.e. as $t \rightarrow \infty$, what are Z and X?

[1] In reality, stiffness K_2 represents the stiffness of the cable wrapped on the pulley. Rarely elements are "rigid"

FREE BODY diagrams and kinematic constraints

Definitions:

F_{s1} = force from elastic cable connected to tree
F_{s2} = force from elastic cable connecting car to cable on pulley = T = Tension on cable
F_D = viscous drag force
F_1 = engine force

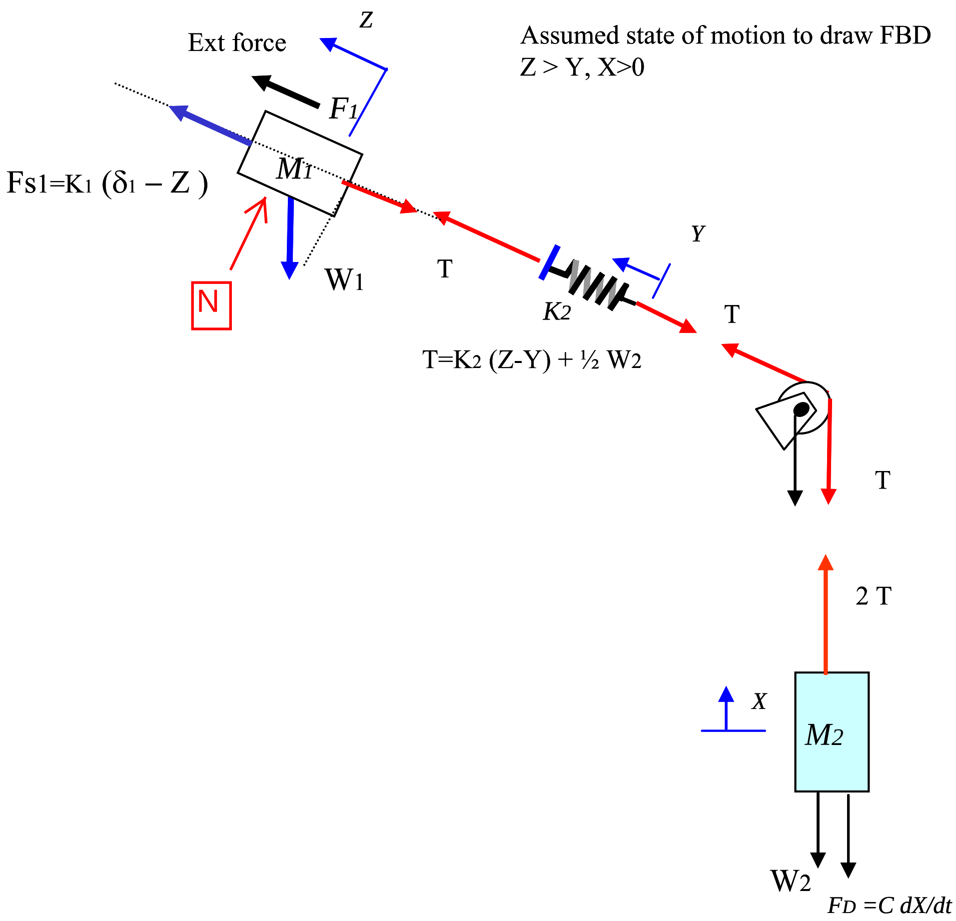
N: normal force (wall)

δ_{s1} , δ_{s2} are static deflections for spring 1 and 2, respectively

Kinematic constraint – inextensible cable

2 T δX = T δY, hence

2 δX = δY



MEEN 363 EXAMPLE 2DOF- ANALYSIS: car pulling a block

ORIGIN := 1

$K_2 := 10^5 \cdot \frac{lb}{in}$ $\quad W_2 := 5000 \cdot lb$ $\quad C := 1500 \cdot lb \cdot \frac{sec}{in}$ $\quad M_2 := \frac{W_2}{g}$ $\quad M_2 = 155.405 \frac{lb\,sec^2}{ft}$

$K_1 := 10^5 \cdot \frac{lb}{in}$ $\quad W_1 := 1000 \cdot lb$ $\quad \theta := 20 \cdot \frac{\pi}{180}$ $\quad M_1 := \frac{W_1}{g}$ $\quad M_1 = 31.081 \frac{lb\,sec^2}{ft}$

masses need be expressed in lb.sec^2/ft for consistency in EOM

(a) kinematic constraint - inextensible cable

The cable length is constant, thus $\quad l_c = l_c + 2 \cdot X - Y \quad$ and the kinematic constraint follows as $\quad Y = 2 \cdot X$

(b) Derive EOMS: Assume a state of motion with Z-Y>0, X>0 motorcar pulls block

Block of mass M2: From the FBD diagram, with X>0, and apply Newton's 2nd law to obtain:

$M_2 \cdot \frac{d^2}{dt^2} X = -W_2 - F_{Damper} + 2 \cdot T$ (1) where $\quad F_{Damper} = C \cdot \frac{d}{dt} X$ (2) is the viscous drag force

$T = \left[K_2 \cdot (Z - Y) + K_2 \cdot \delta_{s2}\right] = F_{s2}$ (3) T is the cable tension = Force from spring 2.
(Z-Y)>0, and δs2 is the static deflection for spring 2

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Car of mass M1: From the FBD diagram, with Z>Y=2X, apply Newton's 2nd law to obtain:

$M_1 \cdot \frac{d^2}{dt^2} Z = -W_1 \cdot \sin(\theta) + F_1(t) + F_{s1} - T$ (4) where F1(t) is engine force

$F_{s1} = K_1 \cdot (\delta_{s1} - Z)$ (5) is the force from cable 1 connected to tree

(c) Before proceeding - find the **springs' static deflection.** For statics, assume NO motion, hence no engine force

and since Z=Y=X=0 means static equilibrium position:

and equations (1) and (4) reduce to $\quad T := \frac{W_2}{2}$ and $\quad T = K_2 \cdot \delta_{s2} = F_{s2}$ $\quad \delta_{s2} := \frac{W_2}{2 \cdot K_2}$ $\quad \delta_{s2} = 0.025\ in$

$F_{s1} = T + W_1 \cdot \sin(\theta)$ and $\quad F_{s1} := W_1 \cdot \sin(\theta) + \frac{W_2}{2}$ $\quad \delta_{s1} := \frac{F_{s1}}{K_1}$ $\quad \delta_{s1} = 0.028\ in$ (6)

Note that the spring attached to the fixed point (a tree for example) must hold 50% of block2 weight and a fraction of the car weight.

It should be easy for you to derive the results in Eq. (6) w/o the aid of the complete equations (1) and (4), i.e. by **simple statics**.

(d) Derive EOMs for the 2-DOF system: Back to DYNAMICS

substitute T and Fdamper into (1) to obtain

$$M_2 \cdot \frac{d^2}{dt^2} X = -W_2 - F_{Damper} + 2 \cdot T \qquad (1)$$

$$M_2 \cdot \frac{d^2}{dt^2} X = -W_2 - C \cdot \frac{d}{dt} X + 2 \cdot \left[K_2 \cdot \left[(Z - 2 \cdot X) + K_2 \cdot \delta_{s2} \right] \right]$$

and since $2 \cdot K_2 \cdot \delta_{s2} = W_2$

$$M_2 \cdot \frac{d^2}{dt^2} X = -C \cdot \frac{d}{dt} X + 2 \cdot \left[K_2 \cdot (Z - 2 \cdot X) \right]$$

BLOCK of MASS M1:

$$M_2 \cdot \frac{d^2}{dt^2} X + C \cdot \frac{d}{dt} X + 4 \cdot K_2 \cdot X - 2 \cdot K_2 \cdot Z = 0 \qquad (7)$$

substitute Fs1 and T into (5)

$$M_1 \cdot \frac{d^2}{dt^2} Z = -W_1 \cdot \sin(\theta) + F_1(t) + F_{s1} - T \qquad (5)$$

$$M_1 \cdot \frac{d^2}{dt^2} Z = -W_1 \cdot \sin(\theta) + F_1(t) + K_1 \cdot \left(\delta_{s1} - Z \right) - \left[K_2 \cdot (Z - Y) + K_2 \cdot \delta_{s2} \right]$$

Sub Y=2X

$$M_1 \cdot \frac{d^2}{dt^2} Z = -W_1 \cdot \sin(\theta) + F_1(t) + K_1 \cdot \left(\delta_{s1} - Z \right) - \left[K_2 \cdot (Z - 2 \cdot X) + K_2 \cdot \delta_{s2} \right]$$

$$M_1 \cdot \frac{d^2}{dt^2} Z = -W_1 \cdot \sin(\theta) + F_1(t) + K_1 \cdot \delta_{s1} - K_1 \cdot Z - K_2 \cdot (Z - 2 \cdot X) - \frac{W_2}{2}$$

from statics:

$$K_1 \cdot \delta_{s1} = \left(W_1 \cdot \sin(\theta) + \frac{W_2}{2} \right) = F_{s1}$$

cancelling terms, i.e. applying statics

$$M_1 \cdot \frac{d^2}{dt^2} Z = F_1(t) - K_1 \cdot Z - K_2 \cdot (Z - 2 \cdot X)$$

BLOCK of MASS M2

$$M_1 \cdot \frac{d^2}{dt^2} Z + \left(K_1 + K_2 \right) \cdot Z - 2 \cdot K_2 \cdot X = F_1(t) \qquad (8)$$

Eqns. (7) and (8) are the desired equations of motion for the cart and block. Please recall that motions are from the static equilibrium position

In matrix form, the EOMs are:

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \cdot \frac{d^2}{dt^2} \begin{pmatrix} Z \\ X \end{pmatrix} + \begin{pmatrix} K_1 + K_2 & -2K_2 \\ -2 \cdot K_2 & 4 \cdot K_2 \end{pmatrix} \cdot \begin{pmatrix} Z \\ X \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \cdot \frac{d}{dt} \begin{pmatrix} Z \\ X \end{pmatrix} = \begin{pmatrix} F_1(t) \\ 0 \end{pmatrix} \qquad (9)$$

Note that mass & stiffness matrices are symmetric.The damping matrix is NOT

(d) Find natural frequencies and natural mode shapes of UNDAMPED system.

Disregading damping, and letting the force F1=0, eq. (9) becomes

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \cdot \frac{d^2}{dt^2}\begin{pmatrix} Z \\ X \end{pmatrix} + \begin{pmatrix} K_1 + K_2 & -2K_2 \\ -2\cdot K_2 & 4\cdot K_2 \end{pmatrix} \cdot \begin{pmatrix} Z \\ X \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (10)$$

Let $Z = a_1 \cdot \cos(\omega \cdot t)$ $\quad X = a_2 \cdot \cos(\omega \cdot t)$ $\quad$ (11)

Substitution of (11) into (10) gives

$$\begin{pmatrix} K_1 + K_2 - M_1 \cdot \omega^2 & -2K_2 \\ -2\cdot K_2 & 4\cdot K_2 - M_2 \cdot \omega^2 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \cos(\omega \cdot t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

cancel cos(ωt) since <> 0 for all times

The homogeneous system of eqns,

$$\begin{pmatrix} K_1 + K_2 - M_1 \cdot \omega^2 & -2K_2 \\ -2\cdot K_2 & 4\cdot K_2 - M_2 \cdot \omega^2 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (12)$$

has a non-trivial solution if the determinant of the system of equations equals zero, i.e. if

$$\Delta(\omega) = \left(K_1 + K_2 - M_1 \cdot \omega^2\right) \cdot \left(4\cdot K_2 - M_2 \cdot \omega^2\right) - 4\cdot K_2^2 = 0$$

Let $\lambda = \omega^2$ and expanding the products in the eqn. above

$$0 = \lambda^2 \cdot M_1 \cdot M_2 - \lambda \cdot \left[\left(K_1 + K_2\right) \cdot M_2 + 4\cdot K_2 \cdot M_1\right] + \left(K_1 + K_2\right) \cdot 4K_2 - 4\cdot K_2^2$$

Let: $0 = \left(a \cdot \lambda^2 + b \cdot \lambda + c\right)$ with:

$$a := M_1 \cdot M_2$$

$$b := -\left[\left(K_1 + K_2\right) \cdot M_2 + 4\cdot K_2 \cdot M_1\right]$$

$$c := \left(K_1 + K_2\right) \cdot 4\cdot K_2 - 4\cdot K_2^2$$

IT IS MOST IMPORTANT HERE TO USE THE RIGHT PHYSICAL UNITS FOR MASS.

$a = 4.83 \times 10^3 \frac{lb^2 \, sec^4}{ft^2}$ $\quad b = -5.222 \times 10^8 \frac{lb^2 \, sec^2}{ft^2}$ $\quad c = 5.76 \times 10^{12} \frac{lb^2}{ft^2}$

NOTE That the coefficients a,b,c have consistent physical units. That is, since the physical unit for λ is (1/sec^2), the physical units of the determinant are (lbf/ft)^2

The roots (eigenvalues) of the characteristic equation are

$$\lambda_1 := \frac{-b - \left(b^2 - 4\cdot a\cdot c\right)^{0.5}}{2\cdot a} \qquad \lambda_2 := \frac{-b + \left(b^2 - 4\cdot a\cdot c\right)^{0.5}}{2\cdot a} \qquad \lambda = \begin{pmatrix} 1.247\times 10^4 \\ 9.564\times 10^4 \end{pmatrix} \frac{1}{sec^2}$$

and the natural frequencies are:

$$\omega_1 := \left(\lambda_1\right)^{0.5} \qquad \omega_2 := \left(\lambda_2\right)^{0.5} \qquad \omega = \begin{pmatrix} 111.666 \\ 309.25 \end{pmatrix} \frac{rad}{sec}$$

Now, find the eigenvectors

The two equations in (12) are linearly dependent. Thus, one cannot solve for a1 and a2. Set arbitrarily; and from the first equation $a_1 := 1$

for ω_1

$$a_2 := \frac{\left[K_1 + K_2 - M_1\cdot\left(\omega_1\right)^2\right]\cdot a_1}{2\cdot K_2}$$

$$\phi_1 := a \qquad \phi_1 = \begin{pmatrix} 1 \\ 0.839 \end{pmatrix}$$

is the first eigenvector (natural mode)

DOF1 (Z) and DOF2 (X) move in phase, with Z>X

for ω_2

$$a_2 := \frac{\left[K_1 + K_2 - M_1\cdot\left(\omega_2\right)^2\right]\cdot a_1}{2\cdot K_2}$$

$$\phi_2 := a \qquad \phi_2 = \begin{pmatrix} 1 \\ -0.239 \end{pmatrix}$$

is the 2nd eigenvector (natural mode)

DOF1 (Z) and DOF2 (X) move OUT of phase, with |Z|>|X|

CHECK: computational software allows quick calculation of eigenvalues and eigenvectors

Define

$$M := \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \qquad K := \begin{pmatrix} K_1 + K_2 & -2K_2 \\ -2\cdot K_2 & 4\cdot K_2 \end{pmatrix}$$

as the mass and stiffness matrices

Let

$$MinvK := M^{-1}\cdot K$$

$$\lambda_C := (eigenvals(MinvK)) \qquad \lambda_C = \begin{pmatrix} 9.564\times 10^4 \\ 1.247\times 10^4 \end{pmatrix} \frac{1}{sec^2}$$

found from highest to lowest

$$\phi_{C_1} := eigenvec\left(MinvK, \lambda_1\right) \qquad \phi_{C_1} = \begin{pmatrix} 0.766 \\ 0.643 \end{pmatrix}$$

$$\phi_{C_2} := eigenvec\left(MinvK, \lambda_2\right) \qquad \phi_{C_2} = \begin{pmatrix} 0.973 \\ -0.232 \end{pmatrix}$$

These eigenvectors are identical to the ones found analytically, since

$$\frac{\left(\phi_{C_1}\right)_2}{\left(\phi_{C_1}\right)_1} = 0.839 \quad \text{and} \quad \frac{\left(\phi_{C_2}\right)_2}{\left(\phi_{C_2}\right)_1} = -0.239$$

That is, the ratio of the second element to first element is fixed. The actual values are not important

(f) Response in MODAL coordinates: Use the transformation $q = A \cdot x$

f.1 Make modal matrix using eigenvectors $A := \text{augment}(\phi_1, \phi_2)$

$$A = \begin{pmatrix} 1 & 1 \\ 0.839 & -0.239 \end{pmatrix}$$

f.2 check orthogonality property of natural modes

$$M_M := A^T \cdot M \cdot A \qquad K_M := A^T \cdot K \cdot A$$

$$M_M = \begin{pmatrix} 140.348 & -3.944 \times 10^{-14} \\ -4.041 \times 10^{-14} & 39.922 \end{pmatrix} \frac{\text{lb sec}^2}{\text{ft}} \qquad K_M = \begin{pmatrix} 1.75 \times 10^{6} & -5.239 \times 10^{-10} \\ -5.107 \times 10^{-10} & 3.818 \times 10^{6} \end{pmatrix} \frac{\text{lb}}{\text{ft}}$$

non-diagonal elements are very small= non zero b/c of roundoff with computer

f.3 define modal masses and stiffnesses:

$$M_{m_1} := M_{M_{1,1}} \qquad M_{m_2} := M_{M_{2,2}}$$

$$K_{m_1} := K_{M_{1,1}} \qquad K_{m_2} := K_{M_{2,2}}$$

check $\left(\frac{K_{m_1}}{M_{m_1}}\right)^{0.5} = 111.666 \frac{\text{rad}}{\text{sec}}$ $\qquad \left(\frac{K_{m_2}}{M_{m_2}}\right)^{0.5} = 309.25 \frac{\text{rad}}{\text{sec}}$ OK

f.3 define initial conditions: displacements and velocities in modal coordinates

At time t=0s, the system is at its static equilibrium position, hence the initial conditions are null displacements and null velocities. Of course, the same applies to modal space, i.e. null initial displacements and velocities

for generality, define: $X_o := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \text{ft}$ Z X $\qquad V_o := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \frac{\text{ft}}{\text{sec}}$

Calculate inverse of A matrix $A_{inv} := A^{-1}$

$$A = \begin{pmatrix} 1 & 1 \\ 0.839 & -0.239 \end{pmatrix}$$

and in modal coordinates

$$q_o := A_{inv} \cdot X_o \qquad q_{o\_dot} := A_{inv} \cdot V_o \qquad q_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ft$$

$$q_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ft \qquad q_{o\_dot} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \frac{ft}{sec}$$

f.4 Define modal force $\quad F_o := 1000 \cdot lb$

$$F := \begin{pmatrix} F_o \\ 0 \cdot lb \end{pmatrix}$$ Physical force vector

$$Q := A^T \cdot F$$

$$Q = \begin{pmatrix} 1 \times 10^3 \\ 1 \times 10^3 \end{pmatrix} lb$$

Both natural modes will be excited

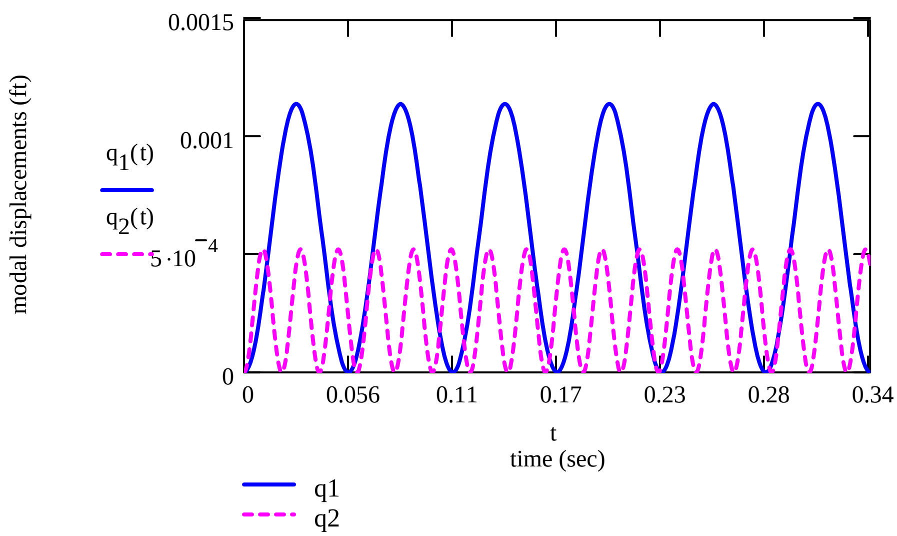
f.5 modal response

Using knowledge to solve $\quad M_{m_i}\left(\frac{d^2}{dt^2}q_i\right) + K_{m_i} \cdot q_i = Q_i \qquad i = 1, 2$

and since the initial conditions are null

$$q_1(t) := \frac{Q_1}{K_{m_1}} \cdot \left(1 - \cos\left(\omega_1 \cdot t\right)\right) \text{ and } \qquad q_2(t) := \frac{Q_2}{K_{m_2}} \cdot \left(1 - \cos\left(\omega_2 \cdot t\right)\right)$$

$$\omega = \begin{pmatrix} 111.666 \\ 309.25 \end{pmatrix} \frac{rad}{sec} \qquad T_{large} := \frac{2 \cdot \pi}{\omega_1} \cdot 6$$ arbitrary scale for plot



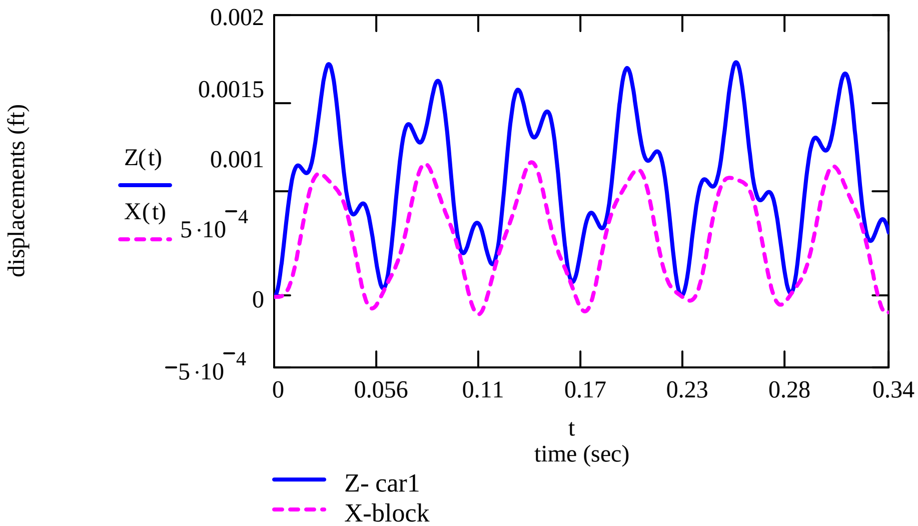
Note that there is no damping or attenuation of motions.

The response in physical coordinates, z and x, equals (from transformation x=Aq)

$$A = \begin{pmatrix} 1 & 1 \\ 0.839 & -0.239 \end{pmatrix}$$

$$Z(t) := q_1(t) + q_2(t)$$

$$X(t) := 0.839 \cdot q_1(t) - 0.239 \cdot q_2(t)$$



The mean value of Z and X are:

$$Z_{mean} := \frac{Q_1}{K_{m_1}} + \frac{Q_2}{K_{m_2}}$$

$$X_{mean} := 0.839 \frac{Q_1}{K_{m_1}} - 0.239 \frac{Q_2}{K_{m_2}}$$

Z_{mean} X_{mean} will be used below

Complicated response with two natural frequencies being excited. However, roughly Z ~ 2 X (as kinematics requires)

(f) FINAL - terminal condition:
With the MATh tools you have, We can NOT solve the problem fully since the damping matrix is NOT proportional. However, damping does exist and aventually the system will achieve a new steady state condition. Since the applied force is a constant,

In the limit as t approaches very, very large values

$$\frac{d^2}{dt^2}\begin{pmatrix} Z \\ X \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} K_1 + K_2 & -2K_2 \\ -2 \cdot K_2 & 4 \cdot K_2 \end{pmatrix} \cdot \begin{pmatrix} Z_{end} \\ X_{end} \end{pmatrix} = \begin{pmatrix} F_o \\ 0 \end{pmatrix}$$

And solving this system of equations using **Cramer's rule**

$$\Delta := \left[\left(K_1 + K_2\right) \cdot 4 \cdot K_2 \right] - 4 \cdot K_2^{\,2} \qquad Z_{end} := \frac{F_o \cdot 4 \cdot K_2}{\Delta} \qquad X_{end} := \frac{2 \cdot K_2 \cdot F_o}{\Delta}$$

$$Z_{end} = 8.333 \times 10^{-4}\,\text{ft} \qquad X_{end} = 4.167 \times 10^{-4}\,\text{ft}$$

$$Z_{mean} = 8.333 \times 10^{-4}\,\text{ft}$$

$$X_{mean} = 4.168 \times 10^{-4}\,\text{ft}$$

$$\frac{Z_{end}}{X_{end}} = 2$$

Note that the graph of undamped periodic motions Z(t) and X(t) shows oscillatory motions abut these terminal or end values.