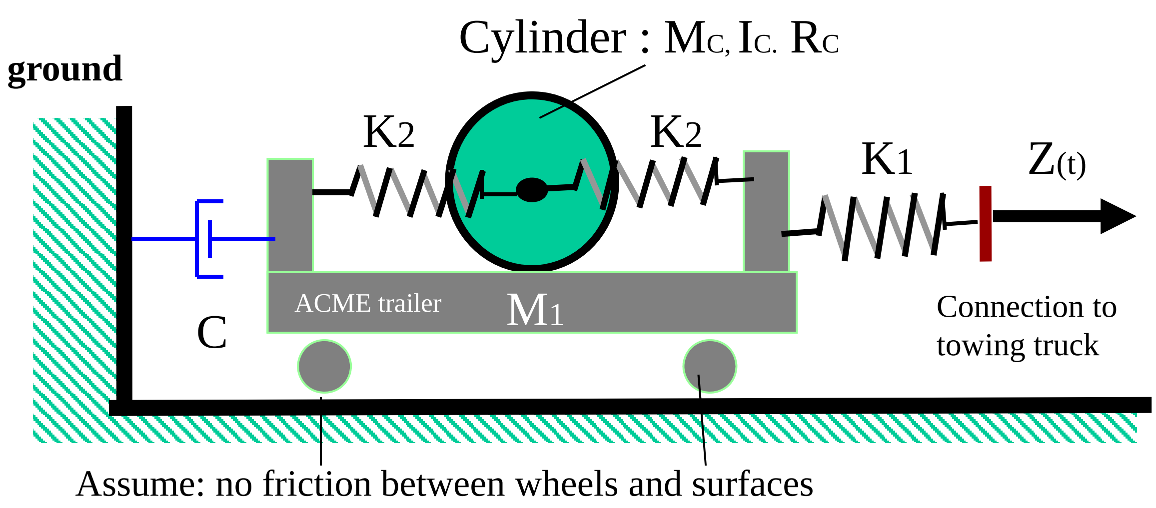
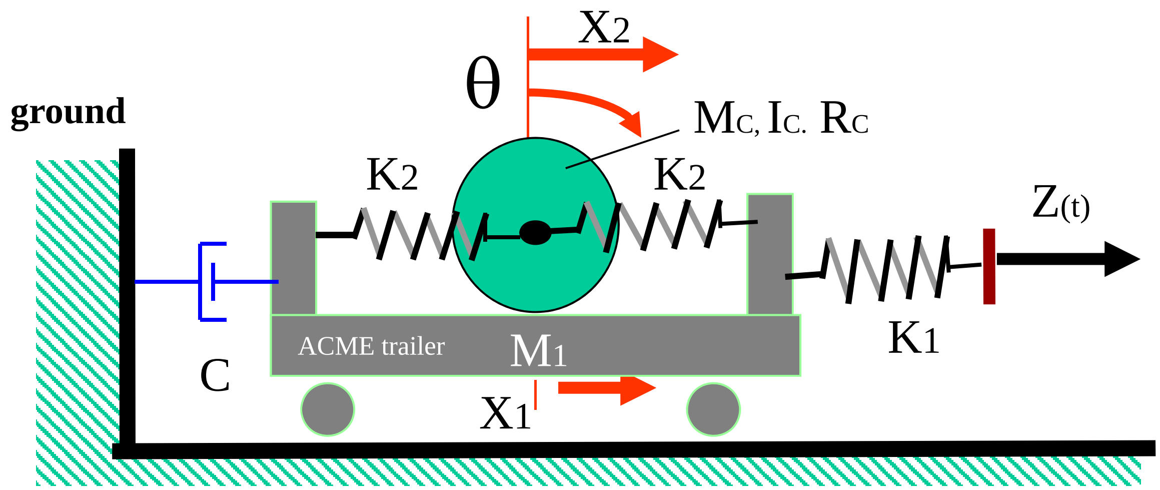
Kinetics of DOF system – DERIVE EOMS

In the figure, cable of stiffness K_1 connects a towing truck (not shown) to a trailer with mass M_1. The trailer transports a cargo cylinder of mass Mc (radius Rc and mass moment of inertia $Ic=M\,a^2\,Rc^2$). The cylinder is held in place with two cables of stiffness K_2 initially stretched with force F_{asy}. The viscous damping coefficient C models a drag or friction force when the trailer is being pulled. The cylinder can roll w/o slipping on the floor of the trailer bed. At t>0 s, the truck applies known displacement Z(t) >0, loading cable (K_1) that drives forward the trailer and its cargo.

Define DOFs and select independent coordinates for the motion of the trailer and cylinder, draw FBDs, and using Newton's Laws (Forces & Moments) derive the system EOMS n of the trailer and its cargo for t>0 s. Express EOMs for the translations of the trailer and cylinder in matrix form [5]

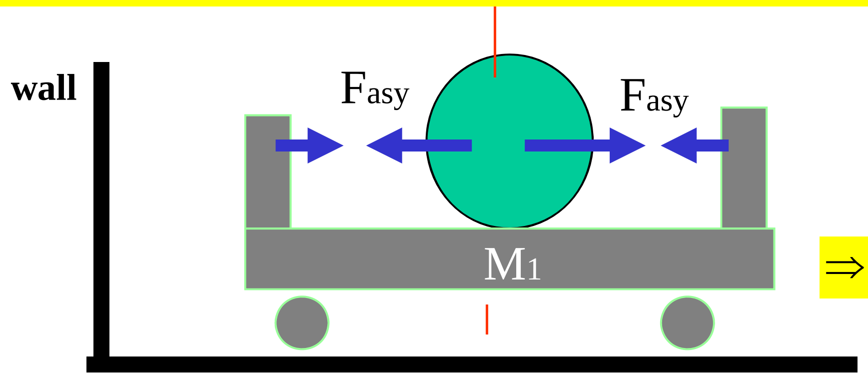




Assumed: no friction between wheels and surfaces

DIAGRAM of forces at STATIC EQUILIBRIUM POSITION

Connection to towing truck



$\Rightarrow \quad \delta_{2L} = -\delta_{2R}$

SEP: $X_1 = X_2 = 0$

$$K_2\delta_{2L} - K_2\delta_{2R} = 0 = F_{asy_L} - F_{asy_R}$$

<u>Notes:</u>

External force not acting, *F(t)*=0

Spring #1 begins to pull on car at *t>0*

DEFINITIONS:

Forces:

F_{asy}: assembly force for springs 2 (extension or stretched)

Parameters:

M_1, M_C: masses trailer & cylinder

K_1, K_2 : stiffness coefficients

C: viscous damping coefficient

$I_C = M_C\, a^2 R_c^2$:
Cylinder mass moment of inertia

Coordinates (Variables):

X_1: translation of trailer

X_2 : translation of cylinder cg

(Absolute frames of reference, with origin at state of rest of trailer and cargo)

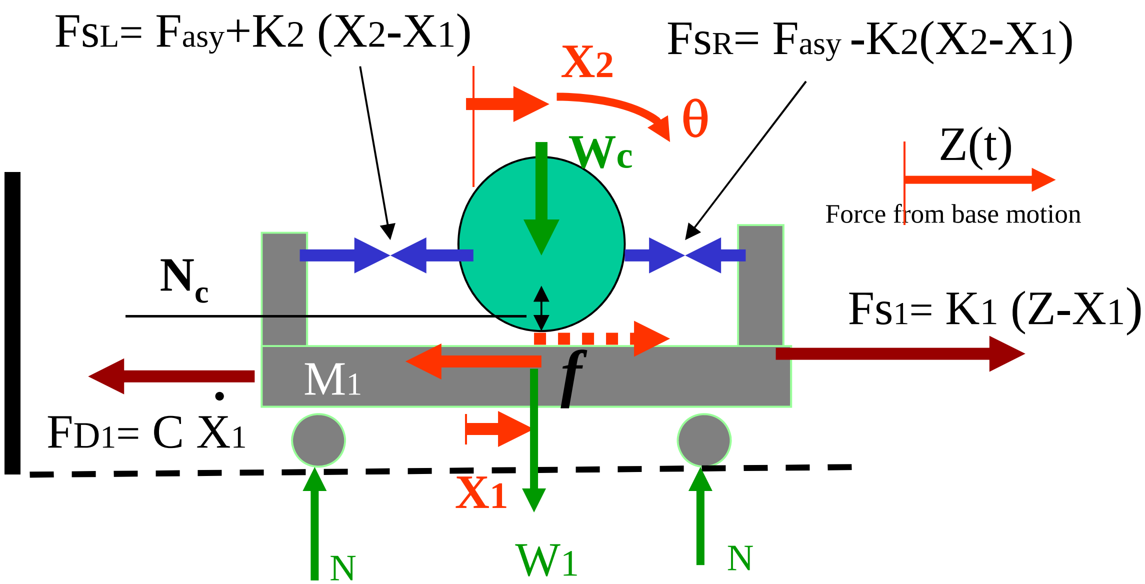
θ Rotation of cylinder,
<u>rolls w/o slipping</u>

$$\theta = \frac{(X_2 - X_1)}{R_C}$$

Z(t) : base motion (known)

Assume a state of motion to draw FBD:

$$X_2 > X_1 > 0,\ Z > X_1 > 0$$



Forces:

W: weight

N: normal force

***f*: contact force - rolling**

Fs: spring force

FD: dashpot force

F_{asy}: assembly force for springs 2 (extension or stretched)

Parameters:

M_1, M_C: masses trailer and cylinder

K_1, K_2: stiffness coefficients

C: viscous damping coefficient

$I_C = M_C\, a^2 R_c^2$:

Cylinder mass moment of inertia

Variables:

X_1: coordinate for motion of trailer

X_2: coordinate for motion of cylinde

(Absolute frames of reference)

Derive Equations of Motion:

STEP 1: State EOMS for each block (mass)

Cylinder

$$M_C \ddot{X}_2 = F_{SR} - F_{SL} + f \quad \textbf{(1a)}$$

$$I_C \ddot{\theta} = -f R_C \quad \textbf{(1b)}$$

Trailer

$$M_1 \ddot{X}_1 = F_{SL} - F_{SR} - f - F_{D1} + F_{S1}(t) \quad \textbf{(2)}$$

Z(t) : base motion (known)

Kinematic constraint $\theta = \dfrac{(X_2 - X_1)}{R_C}$

P1-Derive EOM SP08

Derive Equations of Motion:

STEP 2: Substitute elastic and dashpot forces in EOMS for translation of trailer and cylinder

Cylinder

$$M_C \ddot{X}_2 = F_{asy} - K_2(X_2 - X_1) - F_{asy} - K_2(X_2 - X_1) + f$$

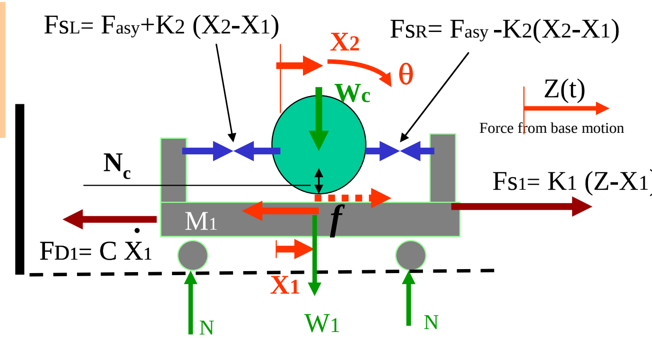
Trailer:

$$M_1 \ddot{X}_1 = F_{asy} + K_2(X_2 - X_1) - F_{asy} + K_2(X_2 - X_1) - f - C\dot{X}_1 + K_1(Z(t) - X_1)$$

(3)

STEP 3: Find contact force from Eq. (1b) and substitute

$$\theta = \frac{(X_2 - X_1)}{R_C}$$



$$-\frac{I_C}{R_C^2}\left(\ddot{X}_2 - \ddot{X}_1\right) = f$$

(4)

STEP 4: Cancel assembly force in Eqs. (3), substitute contact force, and move to LHS terms that depend on motion

Cargo cylinder

$$\left(M_C + \frac{I_C}{R^2}\right)\ddot{X}_2 - \frac{I_C}{R^2}\ddot{X}_1 + 2K_2(X_2 - X_1) = 0$$

(5)

Trailer:

$$M_1 \ddot{X}_1 - \frac{I_C}{R^2}\left(\ddot{X}_2 - \ddot{X}_1\right) + 2K_2(X_1 - X_2) + K_1 X_1 + C\dot{X}_1 = K_1 Z(t)$$

Substitute in Eq. (5)

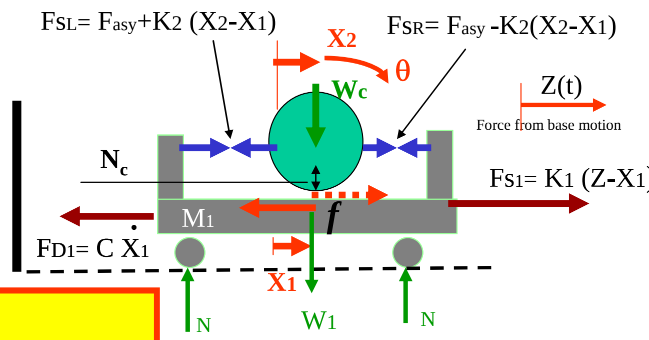
$$I_C = M_C a^2 R_C^2$$

STEP 5: Set EOMs in Matrix Form

$$M_C\left(1+a^2\right)\ddot{X}_2 - M_C\, a^2\ddot{X}_1 + 2K_2\left(X_2 - X_1\right) = 0$$

$$\left(M_1 + M_C a^2\right)\ddot{X}_1 - M_C\, a^2\ddot{X}_2 + 2K_2\left(X_1 - X_2\right) + K_1 X_1 + C\,\dot{X}_1 = K_1\, Z(t)$$

STEP 6: Set EOMs in Matrix Form

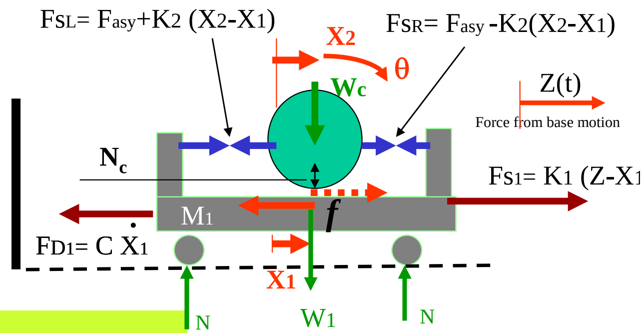


$$\begin{bmatrix} M_C\left(1+a^2\right) & -M_c\, a^2 \\ -M_c\, a^2 & M_1 + M_C\, a^2 \end{bmatrix}\begin{bmatrix} \ddot{X}_2 \\ \ddot{X}_1 \end{bmatrix} +$$

$$\begin{bmatrix} 2K_2 & -2K_2 \\ -2K_2 & K_1 + 2K_2 \end{bmatrix}\begin{bmatrix} X_2 \\ X_1 \end{bmatrix} + \begin{bmatrix} 0 \\ C\dot{X}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ K_1 Z_{(t)} \end{bmatrix}$$

(4)

ENERGIES for system components



Assume a state of motion:

$X_2 > X_1 > 0,\ Z > X_1 > 0$

Kinetic energy = $T = T_{\text{trailer translation}} + T_{\text{Tcylinder – translation}} + T_{\text{cylinder}}$

$$T = \tfrac{1}{2} M_1 \dot{X}_1^2 + \tfrac{1}{2} M_C \dot{X}_2^2 + \tfrac{1}{2} I_C \dot{\theta}^2 = \qquad (1a)$$

$$= \tfrac{1}{2} M_1 \dot{X}_1^2 + \tfrac{1}{2} M_C \dot{X}_2^2 + \tfrac{1}{2} M_C a^2 \left(\dot{X}_2 - \dot{X}_1 \right)^2$$

Potential energy = V = strain energy in cables

$$V = \tfrac{1}{2} K_1 \left(Z - X_1 \right)^2 + \tfrac{1}{2} K_2 \left(X_2 - X_1 + \delta_{2L} \right)^2 + \tfrac{1}{2} K_2 \left(X_1 - X_2 + \delta_{2R} \right)^2 \qquad (1b)$$

$$\delta_{2L} = \delta_{2R}$$

Viscous dissipated power =

$$\wp_v = C \dot{X}_1^2 \qquad (1c)$$

External work $Q=0$

(1d)

No external force applied but a known displacement $Z(t)$

Derive Equations of Motion:

STEP 2: Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} + \frac{1}{2}\frac{\partial \wp_v}{\partial \dot{q}_k} = Q_k \qquad k=1,2\ldots n \qquad (2)$$

STEP 3: Derivatives of Kinetic & potential energies:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_1}\right) = M_1\ddot{X}_1 - M_c\,a^2\left(\ddot{X}_2 - \ddot{X}_1\right); \quad \frac{d}{dt}\left(\frac{\partial T}{\partial X_1}\right) = 0$$

$$\tfrac{1}{2}\frac{\partial \wp_v}{\partial \dot{X}_1} = C\,\dot{X}_1;\ \tfrac{1}{2}\frac{\partial \wp_v}{\partial \dot{X}_2} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_2}\right) = M_C\ddot{X}_2 + M_c\,a^2\left(\ddot{X}_2 - \ddot{X}_1\right); \quad \frac{d}{dt}\left(\frac{\partial T}{\partial X_2}\right) = 0 \qquad (3)$$

$$K_2\delta_{2L} - K_2\delta_{2R} = 0 = F_{asy_L} - F_{asy_R}$$

$$\frac{\partial V}{\partial X_1} = K_1\left(X_1 - Z\right) - K_2\left(X_2 - X_1 + \delta_{2L}\right) + K_2\left(X_1 - X_2 + \delta_{2R}\right)$$
$$= K_1\left(X_1 - Z\right) + K_2\left(2X_1 - 2X_2 + \delta_{2R} - \delta_{2L}\right)$$

$$\frac{\partial V}{\partial X_2} = K_2\left(X_2 - X_1 + \delta_{2L}\right) - K_2\left(X_1 - X_2 + \delta_{2R}\right)$$
$$= K_2\left(2X_2 - 2X_1 + \delta_{2L} - \delta_{2R}\right)$$

STEP 4: Set EOMs in Matrix Form

$$\begin{bmatrix} M_1 + M_c\,a^2 & -M_c\,a^2 \\ -M_c\,a^2 & M_C + M_c\,a^2 \end{bmatrix}\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} +$$
$$\begin{bmatrix} K_1 + 2K_2 & -2K_2 \\ -2K_2 & 2K_2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} C\dot{X}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} K_1 Z_{(t)} \\ 0 \end{bmatrix} \qquad (4)$$