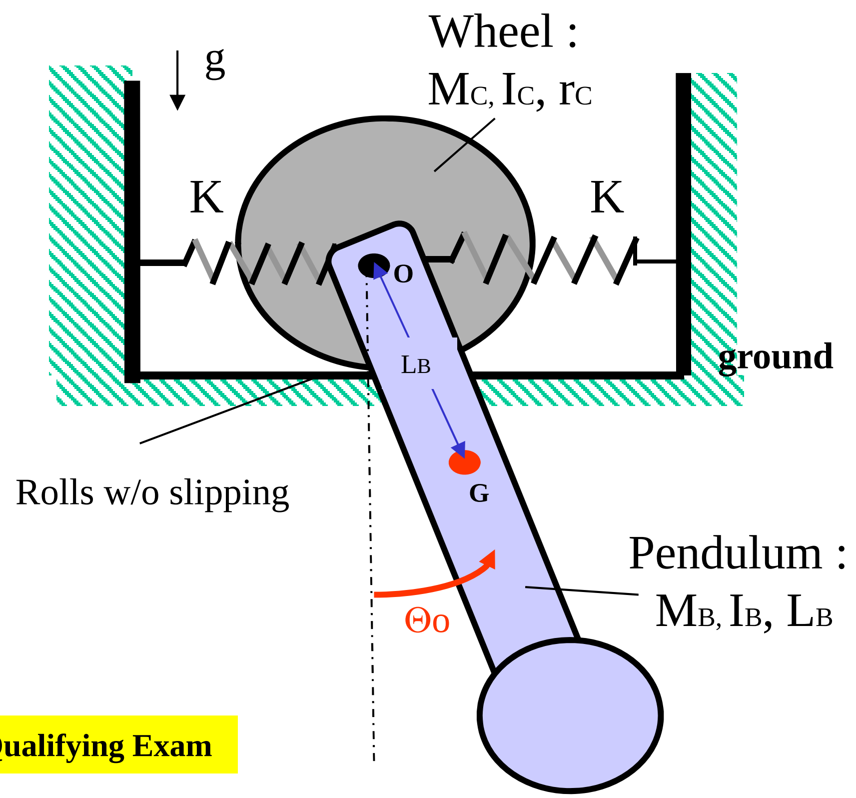
In the figure, two cables of stiffness K connect a wheel of mass M_c to ground. The wheel with radius r_C, has mass moment of inertia is I_C The pendulum, attached to the wheel center, has mass M_B and mass moment of inertia, I_B, with L_B as the distance OG from the pin connection to the pendulum CG. The two cables of stiffness K are initially stretched with force F_{asy}. At time t=0 s, the pendulum is displaced angle Θ_o and released, motion of the whole system follows. Assume there is no friction between the wheels and the horizontal surface.

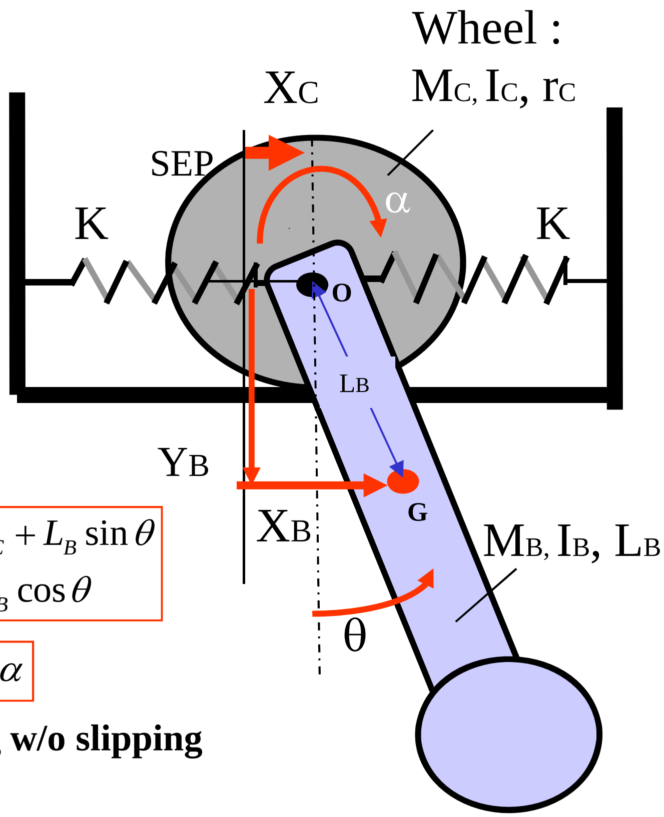
a) Define DOFs, establish kinematic constraints and select independent coordinates for the motion of the wheel and pendulum.

b) draw FBDs and use Newton's Laws (Forces & Moments) to derive the system EOMs of the wheel and the pendulum for t>0 s. **OR**

b) Write system energies and using Lagrangian derive the system EOMs.



SP 2011 Vibrations Qualifying Exam



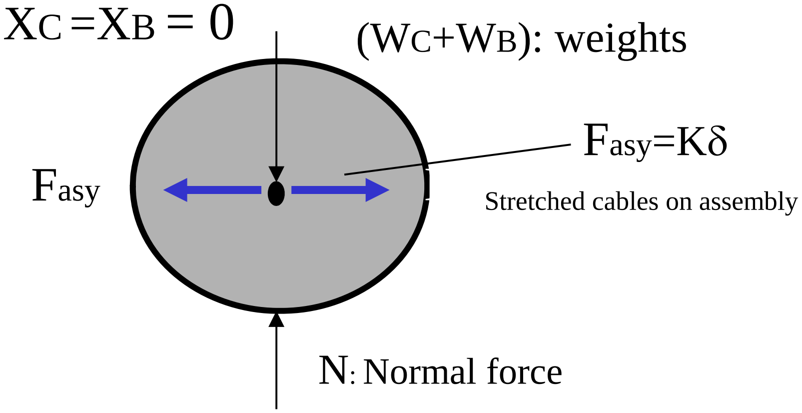
$$\boxed{\begin{aligned} X_B &= X_C + L_B \sin\theta \\ Y_B &= -L_B \cos\theta \end{aligned}}$$

$$\boxed{X_C = r_C\,\alpha}$$

Rolling w/o slipping

DIAGRAM of forces at STATIC EQUILIBLIUM POSITION

SEP: $X_C = X_B = 0$



DEFINITIONS:

Forces:

F_{asy}: assembly force for springs (extension or stretched)

Parameters:

M_B, M_C: masses pendulum & wheel

K : stiffness coefficients

I_B= pendulum mass moment of inertia

I_C= wheel mass moment of inertia

r_c = radius of wheel

L_B = distance O-G

Coordinates (Variables): 5 DOF

X_C: translation of wheel

α: rotation of wheel

$$X_C = r_C\,\alpha \qquad \textbf{(0a)}$$

X_B:Y_B: translation of pendulum cg

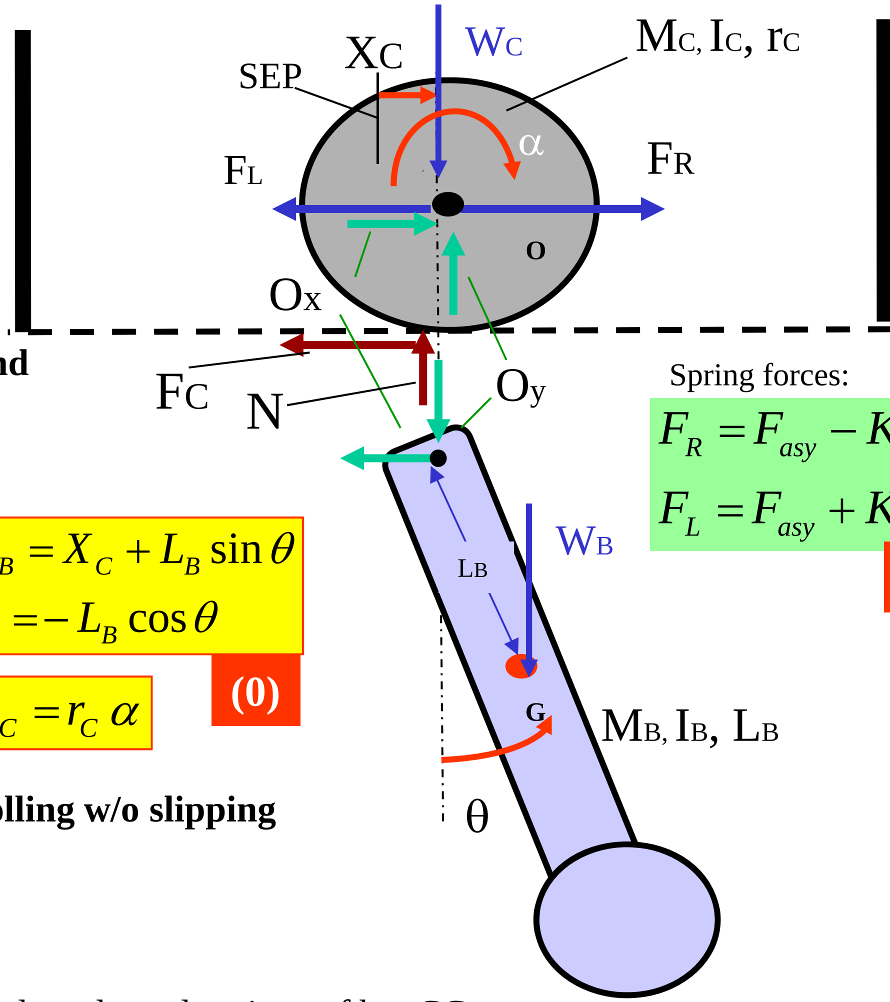
θ: rotation of pendulum,

(Absolute frames of reference, with origin at SEP)

Kinematic constraints:

$$\begin{aligned} X_B &= X_C + L_B \sin\theta \\ Y_B &= -L_B \cos\theta \end{aligned} \qquad \textbf{(0b)}$$

DIAGRAM of forces when moving



Spring forces:

$$F_R = F_{asy} - K\, X_C$$
$$F_L = F_{asy} + K\, X_C$$
(1)

$$X_B = X_C + L_B \sin\theta$$
$$Y_B = -L_B \cos\theta$$
(0)

$$X_C = r_C\, \alpha$$

Rolling w/o slipping

Speeds and accelerations of bar CG

$$\dot{X}_B = \dot{X}_C + L_B\, \dot{\theta} \cos\theta$$
$$\dot{Y}_B = L_B\, \dot{\theta} \sin\theta$$
(2)

$$\ddot{X}_B = \ddot{X}_C + L_B\, \ddot{\theta} \cos\theta - L_B\, \dot{\theta}^2 \sin\theta$$
$$\ddot{Y}_B = L_B\, \ddot{\theta} \sin\theta + L_B\, \dot{\theta}^2 \cos\theta$$
(3)

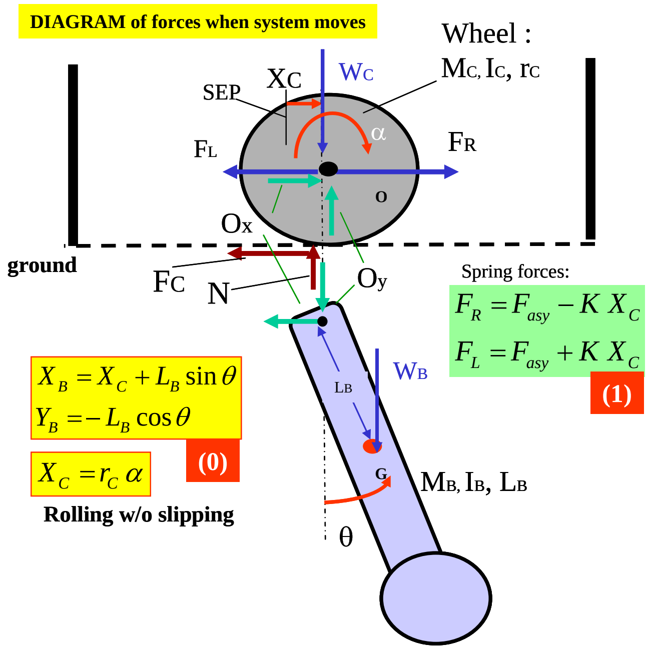
DEFINITIONS:

Forces:
F_L, F_L : forces from springs
O_x, O_y: connection pin forces
$W=Mg$: weight, N: normal force
F_C : contact force

Parameters:
M_B, M_C: masses pendulum & wheel
K : stiffness coefficients
I_B= pendulum mass moment of inertia
I_C= wheel mass moment of inertia
r_c = radius of wheel , L_B = distance O-G

Coordinates (Variables): 5 DOF
X_C: translation of wheel
α: rotation of wheel
X_B:Y_B: translation of pendulum cg
θ: rotation of pendulum

5 DOF – 3 Eqns. Of constraint = 2 independent DOFS



Wheel

$$M_C\,\ddot{X}_C = F_R - F_L + O_x - F_C \qquad \text{(4a)}$$

$$M_C\,\ddot{Y}_C = 0 = -W_C + O_y + N \qquad \text{(4b)}$$

$$I_C\,\ddot{\alpha} = r_C\,F_C \qquad \text{(4c)}$$

Pendulum

$$M_B\,\ddot{X}_B = -O_x \qquad \text{(5a)}$$

$$M_B\,\ddot{Y}_B = -O_y - W_B \qquad \text{(5b)}$$

$$I_B\,\ddot{\theta} = O_x L_B \cos\theta + O_y L_B \sin\theta \qquad \text{(5c)}$$

DEFINITIONS:

Forces:

F_L, F_L : forces from springs

O_x, O_y: connection pin forces

$W=Mg$: weight, N: normal force

F_C : contact force

Parameters:

M_B, M_C: masses pendulum & wheel

K : stiffness coefficients

I_B= pendulum mass moment of inertia

I_C= wheel mass moment of inertia

r_c = radius of wheel , L_B = distance O-G

Coordinates (Variables): 5 DOF

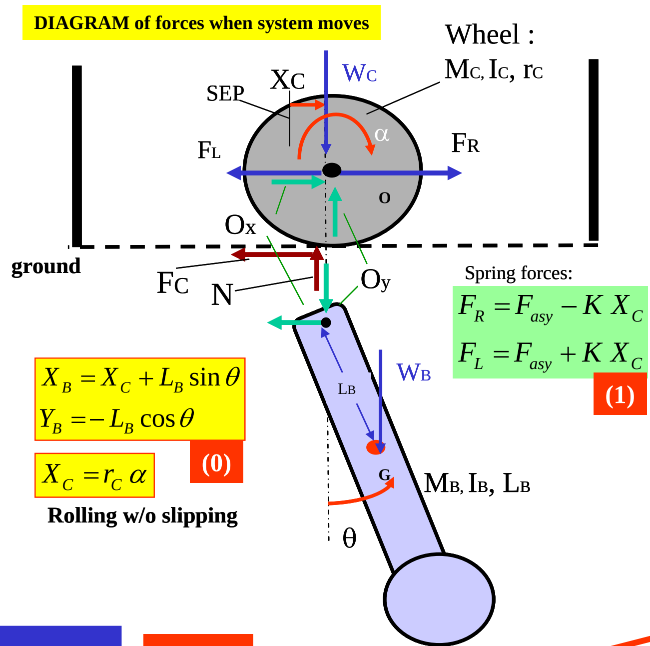
X_C: translation of wheel

α: rotation of wheel

X_B:Y_B: translation of pendulum cg

θ: rotation of pendulum

Select X_C and θ as independent variables



Derive Equations of Motion:

Equations of importance are (4a) and (5c): translation of wheel and rotation of pendulum.

In Eqn (4a): substitute spring forces (2),

$$F_R = F_{asy} - K\,X_C$$
$$F_L = F_{asy} + K\,X_C$$

contact force from (4c),

$$I_C\,\ddot{\alpha} = r_C\,F_C \quad \text{and} \quad \ddot{\alpha} = \frac{\ddot{X}_C}{r_C}$$

and pin force Ox from eqn (5a)

$$M_B\,\ddot{X}_B = -O_x$$

$$\ddot{X}_B = \ddot{X}_C + L_B\,\ddot{\theta}\cos\theta - L_B\,\dot{\theta}^2 \sin\theta$$

Wheel **(4a)**

$$M_C\,\ddot{X}_C = F_R - F_L + O_x - F_C =$$

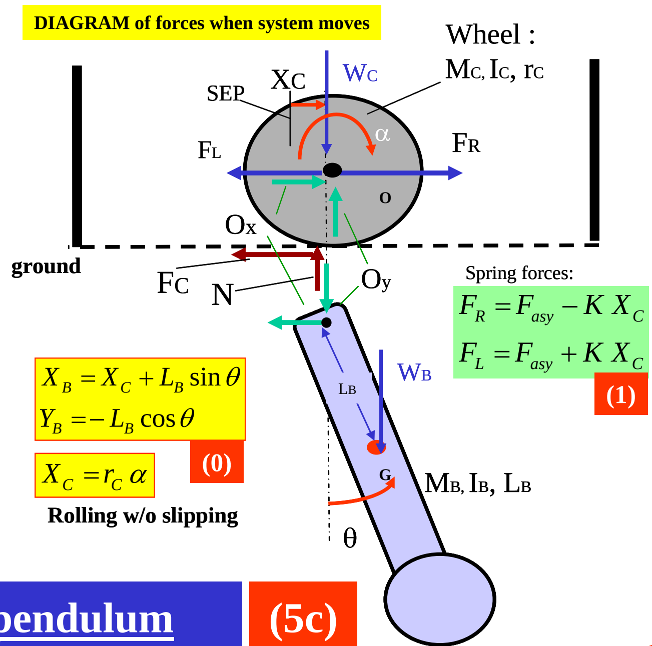
$$= \cancel{F_{asy}} - K\,X_C - \left(\cancel{F_{asy}} + K\,X_C\right) - M_B\,\ddot{X}_B - \frac{I_C}{r_C}\ddot{\alpha}$$

$$= -2\,K\,X_C - M_B\left(\ddot{X}_C + L_B\,\ddot{\theta}\cos\theta - L_B\,\dot{\theta}^2 \sin\theta\right) - \frac{I_C}{r_C^2}\ddot{X}_C$$

Note how assembly (static) forces cancel out.

$$\left(M_C + \frac{I_C}{r_C^2} + M_B\right)\ddot{X}_C + M_B\left(L_B\,\ddot{\theta}\cos\theta - L_B\,\dot{\theta}^2\sin\theta\right) + 2\,K\,X_C = 0 \qquad \textbf{(6)}$$

This is the first EOM describing translation of wheel, rolling w/o slipping, and connected to a swinging pendulum



$$F_R = F_{asy} - K\,X_C$$
$$F_L = F_{asy} + K\,X_C \qquad (1)$$

$$X_B = X_C + L_B \sin\theta$$
$$Y_B = -L_B \cos\theta \qquad (0)$$
$$X_C = r_C\,\alpha$$

Derive Equations of Motion:

In eqn (5c) for rotation of pendulum:

substitute pin reaction forces Ox, Oy from eqns (5a,b)

$$O_x = -M_B\,\ddot{X}_B = -M_B\left(\ddot{X}_C + L_B\,\ddot{\theta}\cos\theta - L_B\,\dot{\theta}^2\sin\theta\right)$$
$$O_y = -M_B\,\ddot{Y}_B - W_B = -M_B\left(L_B\,\ddot{\theta}\sin\theta + L_B\,\dot{\theta}^2\cos\theta\right) - W_B \qquad \textbf{(5a,b)}$$

$$\ddot{X}_B = \ddot{X}_C + L_B\,\ddot{\theta}\cos\theta - L_B\,\dot{\theta}^2\sin\theta$$
$$\ddot{Y}_B = L_B\,\ddot{\theta}\sin\theta + L_B\,\dot{\theta}^2\cos\theta \qquad \textbf{(3)}$$

pendulum **(5c)**

$$\boxed{I_B\,\ddot{\theta} = O_x L_B\cos\theta + O_y L_B\sin\theta}$$
$$= \left(-M_B\left(\ddot{X}_C + L_B\,\ddot{\theta}\cos\theta - L_B\,\dot{\theta}^2\sin\theta\right)\right)L_B\cos\theta$$
$$+\left(-M_B\left(L_B\,\ddot{\theta}\sin\theta + L_B\,\dot{\theta}^2\cos\theta\right) - W_B\right)L_B\sin\theta$$

$$\left(I_B + M_B L_B{}^2\right)\ddot{\theta} + M_B L_B\cos\theta\,\ddot{X}_C + W_B L_B\sin\theta =$$
$$= \left(M_B L_B\,\dot{\theta}^2\cancel{\sin\theta}\right)L_B\cos\theta + \left(-M_B\cancel{L_B}\,\dot{\theta}^2\cos\theta\right)L_B\sin\theta$$

Cancel equal terms

$$\boxed{\left(I_B + M_B L_B{}^2\right)\ddot{\theta} + M_B L_B\cos\theta\,\ddot{X}_C + W_B L_B\sin\theta = 0} \qquad \textbf{(7)}$$

This is the 2nd EOM describing rotation of pendulum, connected to a wheel, rolling w/o slipping

ENERGIES for system components

Kinetic energy = $T = T_{\text{wheel translation}} + T_{\text{wheel rotation}} + T_{\text{pend – translation}} + T_{\text{pend rotation}}$

$$T = \tfrac{1}{2} M_C \, \dot{X}_C^2 + \tfrac{1}{2} I_C \, \dot{\alpha}^2 + \tfrac{1}{2} M_B \left(\dot{X}_B^2 + \dot{Y}_B^2 \right) + \tfrac{1}{2} I_B \, \dot{\theta}^2 =$$

$$= \tfrac{1}{2} \left(M_C + \frac{I_C}{r_C^2} \right) \dot{X}_C^2 + \tfrac{1}{2} M_B \left(\dot{X}_C^2 + 2 \dot{X}_C L_B \, \dot{\theta} \cos\theta + L_B^2 \, \dot{\theta}^2 \right) + \tfrac{1}{2} I_B \, \dot{\theta}^2$$

$$T = \tfrac{1}{2} \left(M_C + \frac{I_C}{r_C^2} + M_B \right) \dot{X}_C^2 + \tfrac{1}{2} M_B \left(2 \dot{X}_C L_B \, \dot{\theta} \cos\theta \right) + \tfrac{1}{2} \left(I_B + M_B L_B^2 \right) \dot{\theta}^2 \qquad (7)$$

Kinematic constraints

$$\boxed{\begin{aligned} \dot{X}_B &= \dot{X}_C + L_B \, \dot{\theta} \cos\theta \\ \dot{Y}_B &= L_B \, \dot{\theta} \sin\theta \\ \dot{\alpha} &= \dot{X}_C / r_C \end{aligned}}$$

Potential energy = $V_{\text{strain energy in cables}} + V_{\text{gravitational from pendulum}}$

$$V = \tfrac{1}{2} K \left(\delta - X_C \right)^2 + \tfrac{1}{2} K \left(\delta + X_C \right)^2 + W_B L_B \left(1 - \cos\theta \right) \qquad (8)$$

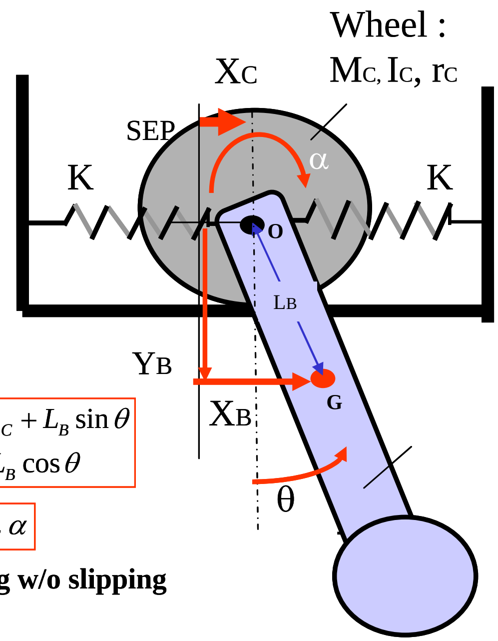
Spring on right spring on left

δ is static deflection (stretching from assembly) = Fasy/K

Viscous dissipated power = $\wp_v = 0$ (9)

External work Q=0 (10)

No external forces applied.
Contact force (rolling w/o slipping) does not perform work



$$\boxed{\begin{aligned} X_B &= X_C + L_B \sin\theta \\ Y_B &= - L_B \cos\theta \end{aligned}}$$

$$\boxed{X_C = r_C \, \alpha}$$

Rolling w/o slipping

Derive Equations of Motion:

STEP 2: Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} + \cancel{\frac{1}{2}\frac{\partial \wp_v}{\partial \dot{q}_k}}^{\,0} = \cancel{Q_k}^{\,0} \qquad k=1,2\ldots n \tag{11}$$

STEP 3: Derivatives of potential energies & kinetic energies

$$\boxed{V = \tfrac{1}{2}K\left(\delta - X_C\right)^2 + \tfrac{1}{2}K\left(\delta + X_C\right)^2 + W_B L_B\left(1-\cos\theta\right)}$$

$$\frac{\partial V}{\partial X_c} = K\left(\delta - X_C\right)(-1) + K\left(\delta + X_C\right) = K\left(-\delta + X_C + \delta + X_C\right) = 2K\,X_C; \tag{12}$$

$$\frac{\partial V}{\partial \theta} = W_B L_B \sin\theta$$

$$\boxed{T = \tfrac{1}{2}\left(M_C + I_C/r_C^2 + M_B\right)\dot{X}_C^2 + \tfrac{1}{2}M_B\left(2\dot{X}_C L_B\,\dot{\theta}\cos\theta\right) + \tfrac{1}{2}\left(I_B + M_B L_B^2\right)\dot{\theta}^2}$$

$$\left(\frac{\partial T}{\partial \dot{X}_c}\right) = \left(M_C + I_C/r_C^2 + M_B\right)\dot{X}_C + M_B\left(L_B\,\dot{\theta}\cos\theta\right);$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_c}\right) = \left(M_C + I_C/r_C^2 + M_B\right)\ddot{X}_C + M_B L_B\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) \qquad \left(\frac{\partial T}{\partial X_c}\right) = 0;$$

$$\left(\frac{\partial T}{\partial \dot{\theta}}\right) = M_B L_B\left(\dot{X}_C\cos\theta\right) + \left(I_B + M_B L_B^2\right)\dot{\theta}; \qquad \left(\frac{\partial T}{\partial \theta}\right) = -M_B L_B\left(\dot{X}_C\,\dot{\theta}\sin\theta\right) \tag{13}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \left(I_B + M_B L_B^2\right)\ddot{\theta} + M_B L_B\left(\ddot{X}_C\cos\theta - \dot{X}_C\,\dot{\theta}\sin\theta\right)$$

Derive Equations of Motion:

$$V=\tfrac{1}{2}K\left(\delta-X_C\right)^2+\tfrac{1}{2}K\left(\delta+X_C\right)^2+W_B L_B\left(1-\cos\theta\right)$$

From:

$$T=\tfrac{1}{2}\left(M_C+{}^{I_C}\!/\!_{r_C^2}+M_B\right)\dot{X}_C^2+\tfrac{1}{2}M_B\left(2\dot{X}_C L_B\dot{\theta}\cos\theta\right)+\tfrac{1}{2}\left(I_B+M_B L_B^2\right)\dot{\theta}^2$$

Equation for translation of wheel

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_C}\right)-\frac{\partial T}{\partial X_C}+\frac{\partial V}{\partial X_C}+0=0$$

$$\left(M_C+{}^{I_C}\!/\!_{r_C^2}+M_B\right)\ddot{X}_C+M_B\left(L_B\ddot{\theta}\cos\theta-L_B\dot{\theta}^2\sin\theta\right)+2K\,X_C=0$$

(14)=(6)

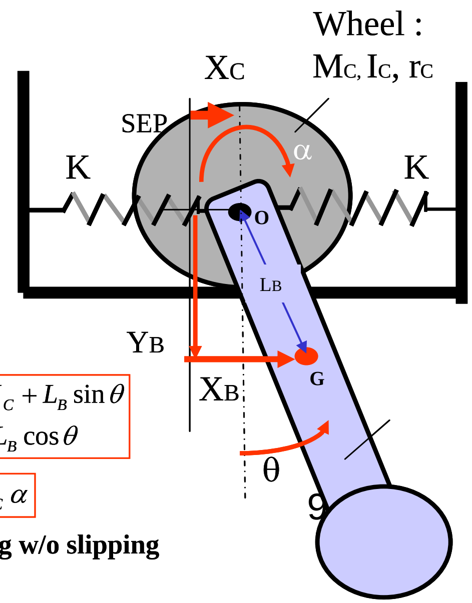
Equation for rotation of pendulum

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)-\frac{\partial T}{\partial \theta}+\frac{\partial V}{\partial \theta}+0=0$$

$$\left(I_B+M_B L_B^2\right)\ddot{\theta}+M_B L_B\left(\ddot{X}_C\cos\theta-\cancel{\dot{X}_C\dot{\theta}\sin\theta}\right)+M_B L_B\left(\cancel{\dot{X}_C\dot{\theta}\sin\theta}\right)+W_B L_B\sin\theta=0$$

$$\left(I_B+M_B L_B^2\right)\ddot{\theta}+M_B L_B\cos\theta\,\ddot{X}_C+W_B L_B\sin\theta=0$$

(15)=(7)



$X_B=X_C+L_B\sin\theta$

$Y_B=-L_B\cos\theta$

$X_C=r_C\,\alpha$

Rolling w/o slipping

Equations of motion (14) and (15) = equations (6) and (7)