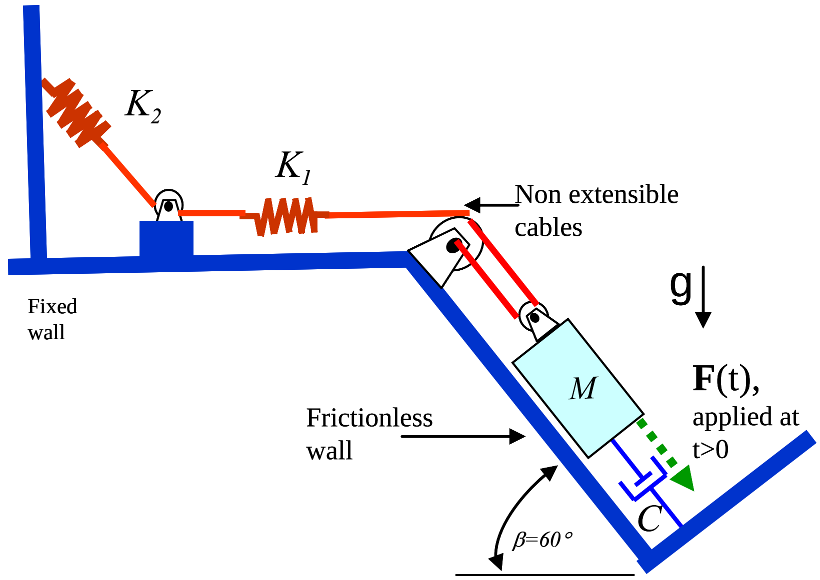
Derive EOM for mechanical system

The figure shows a mechanical system with two elastic cables, represented as stiffnesses K_1 and K_2, connected by a pulley and (inextensible) cables to a heavy block weighing $W=Mg$. The block can slide along the smooth wall (β=60°) . The cables do NOT slip on the pulleys. <u>The system is initially at rest at its **static equilibrium position**</u>.



At time $t>0$ s, external force **F**(t) on the block **drives** the system into motion. A dashpot with damping coefficient C drags the block motion.

a) Select suitable coordinates for motion of the block and springs, show them on the Figure, and **explain** rationale for your choice. Identify the kinematic constraint relating motions of the block and spring 1.

b) Find the <u>**static**</u> deflection (δ_s) of each spring element

c) Draw free body diagrams applicable for time $t>0$ s. Label all forces and relate them to the motion coordinates.

d) Using Netwon's Laws, derive the EOM for the block ($t>0$ s). Use motion of block as independent coordinate

e) Given $K_1=10^5$ lb/in, $K_2=2K_1$, $W=5000$ lb and C= 1500 lb-sec/in, β=60° . Find the system natural frequency [Hz] and viscous damping ratio (ζ).

Explain your solution procedure, detail assumptions. No partial credit for incorrect use of physical units.

P3 SP08 - Derive EOM for simple mechanical system

L San Andres (c) 2008

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b) Find the static deflection (s) of each spring element

c) Draw free body diagrams applicable for time t>0 s. Label all forces and relate them to the motion coordinates.

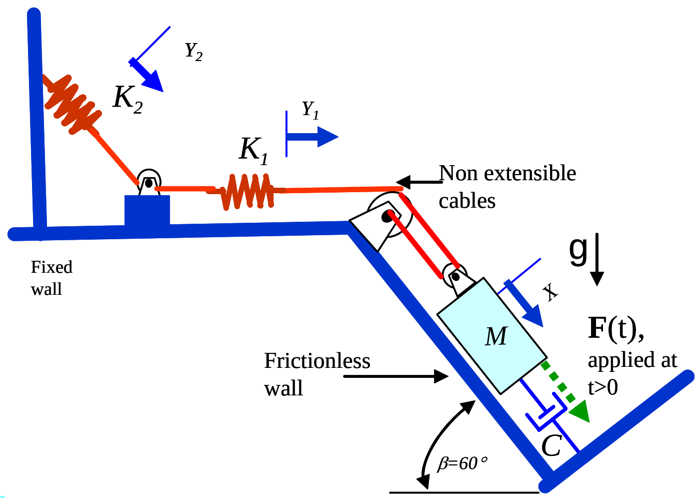
d) Using Netwon's Laws, derive the EOM for the block (t>0 s). Use as independent coordinate motion of block.

e) Find the system natural frequency [Hz] and viscous damping ratio (ζ).

$$W := 5000 \cdot lb \qquad K_1 := 10^5 \cdot \frac{lb}{in}$$

$$M := \frac{W}{g} \qquad K_2 := 2 \cdot K_1$$

$$\beta := \frac{\pi}{3} \qquad C := 1500 \cdot lb \cdot \frac{sec}{in}$$



(a) coordinate system and kinematic constraint - inextensible cable

Static equilibrium position defines origin of coordinates *X*, Y_1 and Y_2 describing the motion of block and springs 1 and 2, respectively.

The cable length is constant, thus

$$l_c = l_c + 2 \cdot x - y_1$$

and the kinematic constraint follows as $y_1 = 2 \cdot x$ **(1)**

(b) Static deflection of springs

By definition of SEP (Static equilibrium position), i.e. when X=Y1=Y2=0 and at rest (without motion):

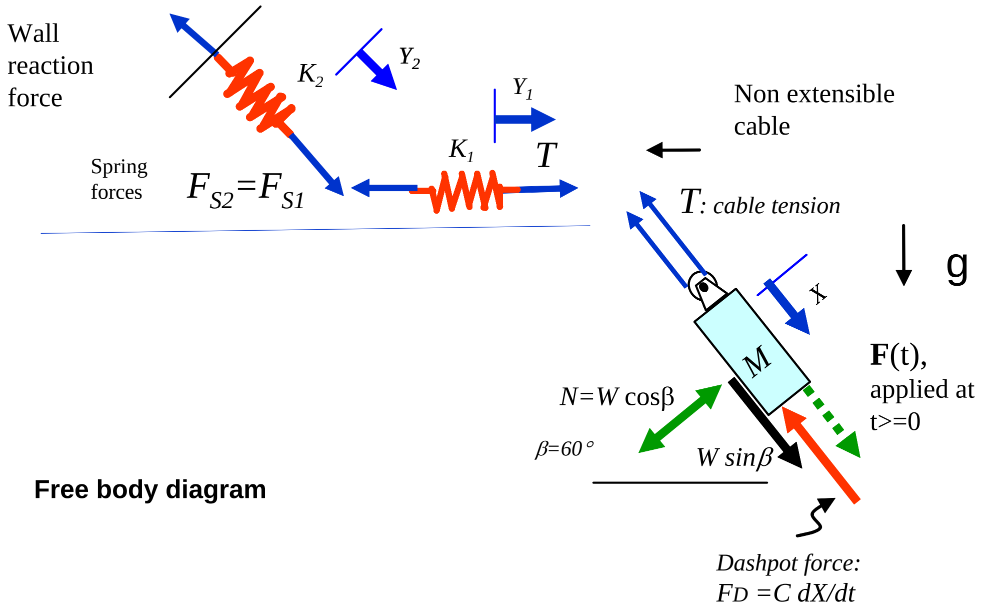
$$0 = -W \cdot \sin(\beta) + 2 \cdot T \quad \& \quad T = F_{s1\_} = F_{s2\_} \qquad T := \frac{W \cdot \sin(\beta)}{2} \qquad \textbf{(2)}$$

Fs_ is the static force from springs needed to HOLD the block w/o motion

Static deflection of the springs are: $\delta_{s1} := \frac{T}{K_1}$ $\quad \delta_{s2} := \frac{T}{K_2}$ **(3)**

$\delta_{s1} = 0.022\, in$ $\quad \delta_{s2} = 0.011\, in$

(c) Assume a state of motion with x>0, y1>0 , y2>0



Free body diagram

From the FBD diagram, Newton's 2nd law gives:

$$M\cdot\frac{d^2}{dt^2}x = W\cdot\sin(\beta) - F_{Damper} - 2\cdot T + F(t) \qquad \textbf{(4)}$$

where $F_{Damper} = C\cdot\frac{d}{dt}x$ (5) is the viscous drag force $N = W\cdot\cos(\beta)$ wall reaction force

$T = F_{S1} = F_{S2}$ (6) Tension = Elastic reaction forces in springs, given by

$$F_{S1} = K_1\cdot(y_1 - y_2) + \frac{W\cdot\sin(\beta)}{2} = F_{S2} = K_2\cdot y_2 + \frac{W\cdot\sin(\beta)}{2} \qquad \textbf{(7)}$$

$$K_1\cdot(y_1 - y_2) = K_2\cdot y_2 = K_S\cdot y_1$$

$K_1\cdot y_1 = (K_2 + K_1)\cdot y_2$ Then $y_2 = \frac{K_1}{(K_2 + K_1)}\cdot y_1$

$$K_1\cdot(y_1 - y_2) = K_1\cdot\left(1 - \frac{K_1}{K_2 + K_1}\right)\cdot y_1$$

$$= \frac{K_1\cdot K_2}{K_1 + K_2}\cdot y_1 = K_S\cdot y_1$$

Thus, the equivalent stiffness equals

$$K_S := \frac{K_1\cdot K_2}{K_1 + K_2} \qquad \textbf{(8)}$$

That is, SPRINGS are in SERIES

$$K_S = 6.667\times 10^4\,\frac{lb}{in}$$

Substitute Eq. (8) into the (7) = (6)=T (Tension) to obtain

$$T = K_S \cdot y_1 + \frac{W \cdot \sin(\beta)}{2} \quad \textbf{(9)}$$

Of course, one could also use the (prior) knowledge of SPRINGS IN SERIES and state directly equation (9) above

(d) Derive single EOM for block motion

Note: EOM cannot contain internal forces (Tension for example). The tension is DETERMINED by the motion.

$$M \cdot \frac{d^2}{dt^2}x = W \cdot \sin(\beta) - C \cdot \frac{d}{dt}x - (2 \cdot K_S \cdot y_1) - W \cdot \sin(\beta) + F(t)$$

and substituting the constraint

$$y_1 = 2 \cdot x$$

$$K_{eq} := 4 \cdot K_S$$

$$K_{eq} = 2.667 \times 10^5 \frac{lb}{in}$$

and thus the final EOM is:

$$M \cdot \frac{d^2}{dt^2}x + C \cdot \frac{d}{dt}x + K_{eq} \cdot x = F(t) \quad \textbf{(10)}$$

(e) Calculate natural frequency and viscous damping ratio:

$$\omega_n := \left(\frac{K_{eq}}{W} \cdot g\right)^{.5}$$

$$\omega_n = 143.497 \frac{1}{sec}$$

$$f_n := \frac{\omega_n}{2 \cdot \pi}$$

$$f_n = 22.838 \text{ Hz}$$

$$\zeta := \frac{C}{2 \cdot \left(K_{eq} \cdot \frac{W}{g}\right)^{.5}}$$

$$\zeta = 0.404$$

$$T_n := \frac{1}{f_n}$$

$$\omega_d := \omega_n \cdot \left(1 - \zeta^2\right)^{0.5}$$

$$\omega_d = 131.291 \frac{rad}{sec}$$

$$f_d := \frac{\omega_d}{2 \cdot \pi}$$

The damping ratio is rather large - motion will be oscillatory but quickly damped!

The damped natural frequency and period of motion are:

$$f_d = 20.896 \text{ Hz}$$

$$T_d = 0.048 \text{ sec}$$

$$T_d := \frac{1}{f_d}$$