MEEN 363. EXAMPLE of ANALYSIS (1 DOF)

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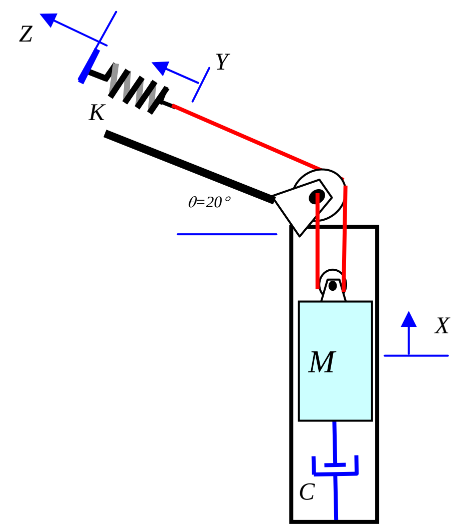
Objectives:

a) To derive EOM for a 1-DOF (one degree of freedom) system
b) To understand concept of static equilibrium
c) To learn the correct usage of physical units (US system)
d) To calculate natural frequency and damping ratio
e) To predict the time response (steady state and full transient) of a system
f) To learn how to combine mathematical statements with explanatory sentences.

Problem statement

A pulley and cable system are assembled to pull a heavy block "*stuck*" in a hollow mining shaft. A motorcar imposes the **known** motion *Z(t)* required to pull the block. The stiffness *K* represents a flexible connection to the drive motorcar. The damping coefficient *(C)* represents the viscous drag between the block and shaft walls.



In the figure, ***Y=X=Z=0*** denote the **static equilibrium position** (SEP) of the system.

a) Draw free body diagrams for the block, label all forces and show their constitutive relation in terms of the motion coordinates, if applicable.
b) Identify the kinematical constraint relating motions *Y* and *X*. The cable does NOT slip on the pulley.
c) Find the **static** deflection (δ_S) of the spring element.
d) Derive a single EOM for the block motion in terms of coordinate *X*.

For items (e) - (h) use $K = 10^5$ lb/in, Mg=5000 lb, and C=1500 lb.s/in

e) Find the system natural frequency [Hz] and viscous damping ratio (ζ).
f) The motorcar moves with $Z(t)=v\,t$, where v= 1 ft/sec. What is the terminal or final velocity of the block (M) in (ft/sec)?
g) Find the complete solution to the problem. That is find *X(t)*

STATIC EQUILIBRIUM POSITION (SEP)

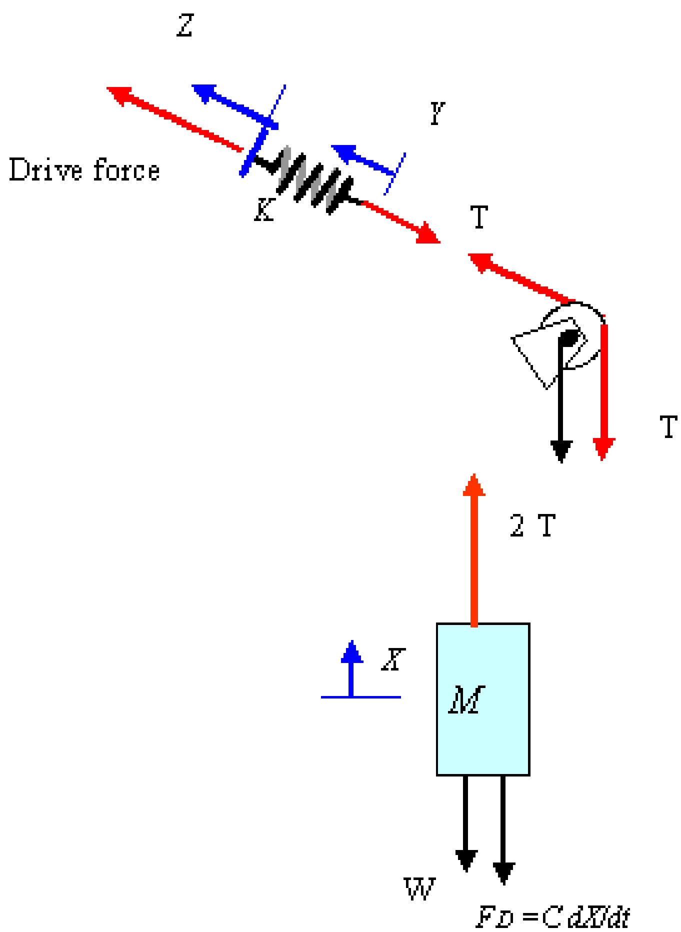
SEP means **no motion** of block or external agent holding spring - cable. Thus, at the SEP spring K is already deflected since it must support 50% of the block weight (W) as easily seen from the cable & pulley constraint. This knowledge is BASIC, does not require of elaborate thinking or deriving lengthy equations.

Important: Z(t) is KNOWN for all times, i.e. a function of time imposed on the system by an external agent (motorcar)

FREE BODY diagrams and kinematic constraints

Definitions:
T = Tension from cable connecting block to spring. Cable is NOT extensible
F_s = force in spring connecting external DRIVE agent to inextensible cable
F_D = viscous drag force
δ_s = static deflection for spring

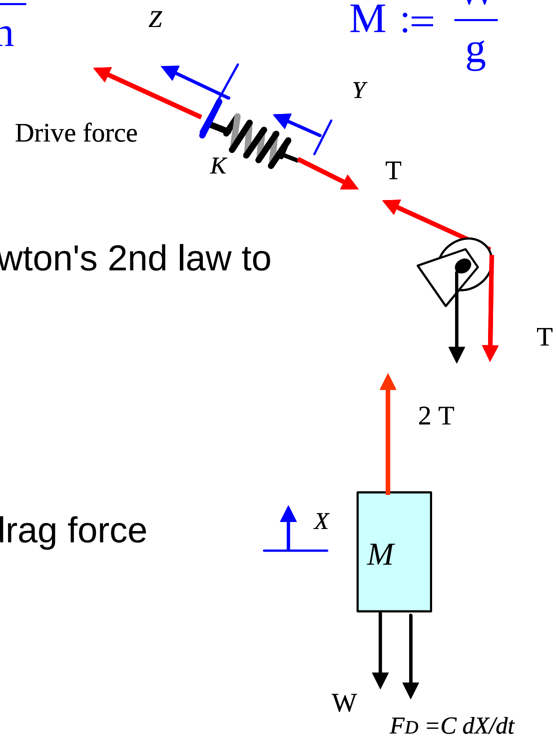
To draw FBDs, assume a state of motion ***X>0, (Z-Y)>0.*** These mathematical statements mean: block moves UP, and spring is transmitting a force also upwards and to the left, see diagrams



where $\mathbf{F_{drive} = F_{spring} = T = K\,\delta_s + K\,(Z\text{-}Y)}$

Note that $\mathbf{(K\,\delta_s = W/2\,)}$ **is the static force in the spring. This the static force necessary to hold the system statically, i.e. without motion. Hence** $\mathbf{\delta_s = 0.5\ W/K}$

$W := 5000 \cdot lb$ $K := 10^5 \cdot \frac{lb}{in}$ $C := 1500 \cdot lb \cdot \frac{sec}{in}$ $M := \frac{W}{g}$



(a) Assume a state of motion with Z-Y>0, X>0 ie. motorcar pulls block

From the FBD diagram, assume X>0, and apply Newton's 2nd law to obtain:

$$M \cdot \frac{d^2}{dt^2} X = -W - F_{Damper} + 2 \cdot T \quad \textbf{(1)}$$

where $F_{Damper} = C \cdot \frac{d}{dt} X$ (2) is the viscous drag force

$$T = \left[K \cdot (Z - Y) + K \cdot \delta_s \right] = F_{Drive} \quad \textbf{(3)}$$

T is the cable tension. (Z-Y)>0, and δs is the spring static deflection

(b) kinematic constraint - inextensible cable

The cable length is constant, thus $l_c = l_c + 2 \cdot X - Y$

(4)

and the kinematic constraint follows as $Y = 2 \cdot X$

(c) Static deflection of spring

By definition of SEP (Static equilibrium position), i.e. when X=Z=Y=0 and at rest (without motion):

$$0 = -W + 2 \cdot K \cdot \delta_s \quad \textbf{(5)}$$

(2K δs) is the static force needed to HOLD the block w/o motion

Static deflection of the spring is: $\delta_s := \frac{W}{2 \cdot K}$ $\delta_s = 0.025\,in$

(c) Derive single EOM for block motion

Note: EOM cannot contain internal forces (Tension for example). The tension is DETERMINED by the motion.

Substituting (2), (3) and (4) into EOM (1) renders

$$M \cdot \frac{d^2}{dt^2} X = -W - C \cdot \frac{d}{dt} X + 2 \cdot \left[K \cdot \left[(Z - 2 \cdot X) + K \cdot \delta_s \right] \right] \quad \textbf{(6)}$$

and thus the final EOM is:

$$M\cdot\frac{d^2}{dt^2}X + C\cdot\frac{d}{dt}X + 4\cdot K\cdot X = 2\cdot K\cdot Z(t) = F(t) \qquad \textbf{(7)}$$

Note $2\cdot K\cdot Z(t)$ **"appears" as a (dynamic) external force driving the block into motion.**

(e) Calculate natural frequency and viscous damping ratio:

$$\omega_n := \left(\frac{4K}{W}\cdot g\right)^{.5}$$

$$\omega_n = 175.747\,\frac{1}{sec} \qquad f_n := \frac{\omega_n}{2\cdot\pi} \qquad f_n = 27.971\,Hz$$

$$\zeta := \frac{C}{2\cdot\left(4\cdot K\cdot\frac{W}{g}\right)^{.5}} \qquad \zeta = 0.33 \qquad T_n := \frac{1}{f_n}$$

$$\omega_d := \omega_n\cdot\left(1-\zeta^2\right)^{0.5} \qquad \omega_d = 165.931\,\frac{rad}{sec} \qquad f_d := \frac{\omega_d}{2\cdot\pi}$$

The damping ratio is rather large - motion will be oscillatory but quickly damped!

$$T_d := \frac{1}{f_d}$$

The damped natural frequency and period of motion are:

$$f_d = 26.409\,Hz \qquad T_d = 0.038\,sec$$

(f) pulling motocar moves with constant speed v, FIND terminal speed of block

Let $v := 1 \cdot \frac{ft}{sec}$ Hence: $z(t) := v \cdot t$ **(8)**

$$M \cdot \frac{d^2}{dt^2} X + C \cdot \frac{d}{dt} X + 4 \cdot K \cdot X = 2 \cdot K \cdot z(t) = F(t) \quad \textbf{(7)}$$

What is steady-state motion?

Since z(t) is linear in time, the particular solution to eqn (7) is:

$$X_p = a + b \cdot t \quad \textbf{(8)}$$

i.e block ALSO moves with constant SPEED $\frac{d}{dt} X_p = b$ $\frac{d^2}{dt^2} X_p = 0$

To find the end or terminal velocity, take the time derivative of (7)

$$M \cdot \frac{d^3}{dt^3} X_p + C \cdot \frac{d^2}{dt^2} X_p + 4 \cdot K \cdot \frac{d}{dt} X_p = 2 \cdot K \cdot v$$

Using the knowledge from derivatives of Xp $\left(4 \cdot K \cdot \frac{d}{dt} X_p\right) = 2K \cdot v$

Terminal velocity of block: $\frac{d}{dt} X_p = \frac{v}{2}$ $\frac{v}{2} = 0.5 \frac{ft}{sec}$

<u>A more elaborate way</u> follows from finding the whole particular solution:

Substitute (8) into EOM (7) to find $C \cdot b + 4 \cdot K(a + b \cdot t) = 2 \cdot K \cdot v \cdot t$

equating like-powers of t $C \cdot b + 2 \cdot K \cdot a = 0$

$4 \cdot K \cdot b = 2 \cdot K \cdot v$

Hence: $b := \frac{v}{2}$ $b = 6 \frac{in}{sec}$

$a := \frac{-C \cdot b}{2 \cdot K}$ $a = -0.045\, in$

From: $X_p(t) := a + b \cdot t$

terminal velocity at which block moves is $b = 0.5 \frac{ft}{sec}$

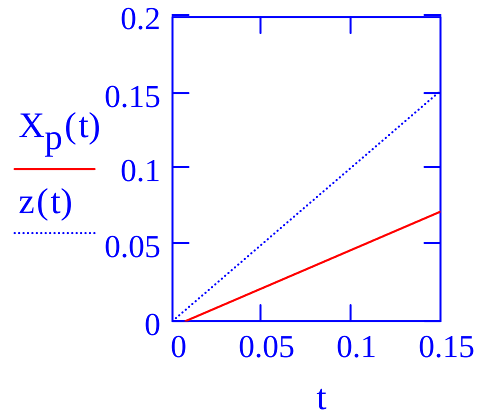
(g) find the full transient response - dynamic motion of block

Particular solution:

$$M \cdot \frac{d^2}{dt^2} X_P + C \cdot \frac{d}{dt} X_P + K \cdot X_P = A + B \cdot t$$

$$X_p = a + b \cdot t$$

$$a = \left(A - C \cdot \frac{B}{K}\right) \cdot \frac{1}{K} \qquad b = \frac{B}{K}$$



Complete solution:

$$X(t) = X_H + X_P$$

$$X(t) = e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(C_1 \cdot \cos\left(\omega_d \cdot t\right) + C_2 \cdot \sin\left(\omega_d \cdot t\right)\right) + (a + b \cdot t) \qquad \textbf{(11)}$$

$$\frac{d}{dt} X = e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(D_1 \cdot \cos\left(\omega_d \cdot t\right) + D_2 \cdot \sin\left(\omega_d \cdot t\right)\right) + b \qquad \textbf{(12)}$$

Where $D_1 = -\zeta \cdot \omega_n \cdot C_1 + C_2 \cdot \omega_d$ $\qquad D_2 = -\zeta \cdot \omega_n \cdot C_2 - C_2 \cdot \omega_d$ **(13)**

satisfy initial conditions at t=0:

$$X_o := 0 \cdot ft \quad V_o := 0 \cdot \frac{ft}{sec}$$

motion starts from rest

from (11) and (12) at time t=0 sec

$$X_o = C_1 + a \qquad C_1 := X_o - a$$

$$V_o = D_1 + b \qquad D_1 := V_o - b$$

and from (13) $\quad C_2 := \dfrac{D_1 + \zeta \cdot \omega_n \cdot C_1}{\omega_d} \qquad D_2 := -\zeta \cdot \omega_n \cdot C_2 - C_2 \cdot \omega_d$

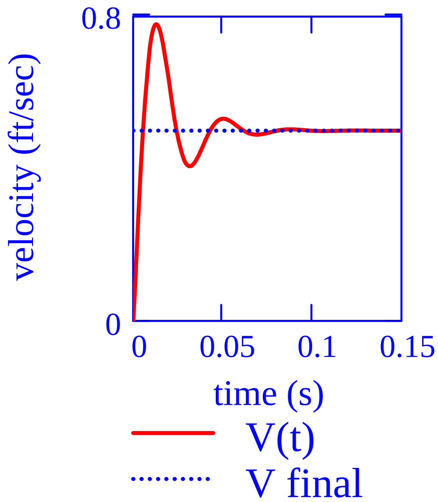
$C_1 = 0.045\ in$ $\qquad C_2 = -0.02\ in$ $\qquad D_1 = -6\ \frac{in}{sec}$ $\qquad D_2 = 4.578\ \frac{in}{sec}$

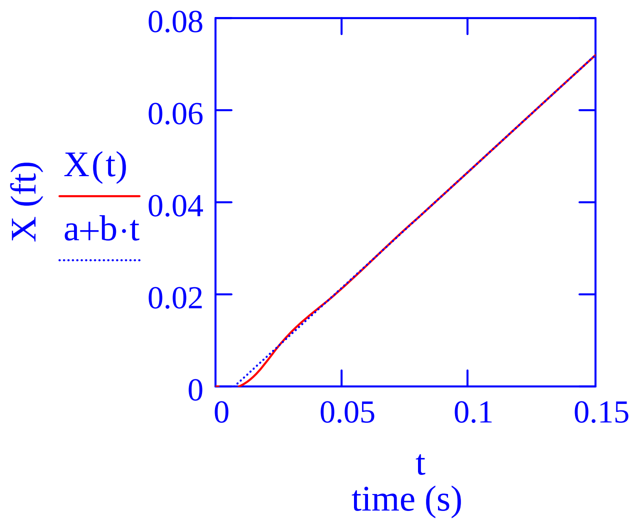
Let's graph the response for time values up to 4 x damped period (my choice)

$$X(t) := e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(C_1 \cdot \cos\left(\omega_d \cdot t\right) + C_2 \cdot \sin\left(\omega_d \cdot t\right)\right) + (a + b \cdot t)$$

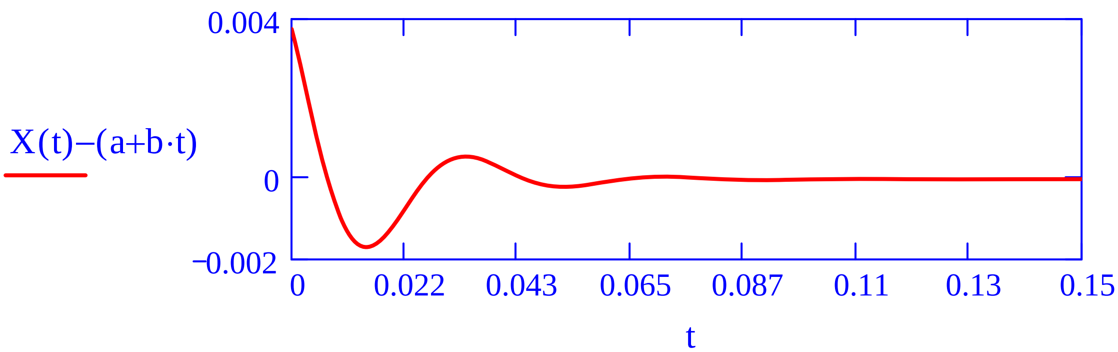
$$V(t) := e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(D_1 \cdot \cos\left(\omega_d \cdot t\right) + D_2 \cdot \sin\left(\omega_d \cdot t\right)\right) + b$$

$$T_{max} := 4 \cdot T_d$$





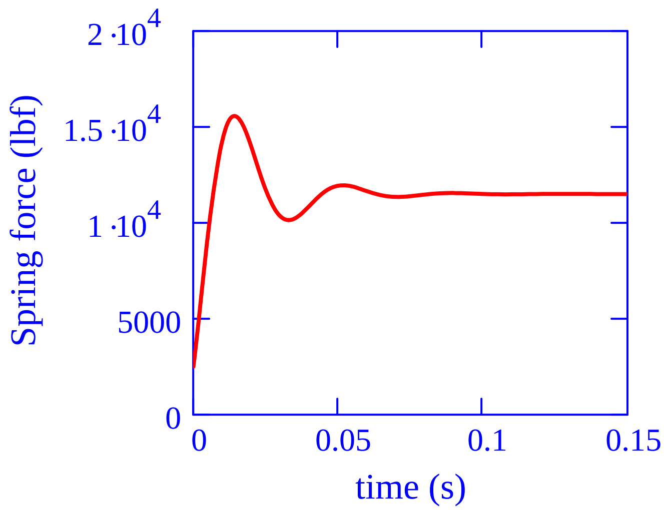
X motion - relative to steady state



Spring (cable) force (dynamic+static)

$$F_s(t) := K\cdot(z(t) - 2\cdot X(t)) + K\cdot\delta_s$$

$W = 5 \times 10^3\,lb$ $K\cdot\delta_s = 2.5 \times 10^3\,lb$



$F_s(T_{max}) = 1.15 \times 10^4\,lb$

at steady state, spring (cable) force approaches

$$F_{SS}(t) := K\cdot\left[v\cdot t - 2\cdot(a + b\cdot t) + \delta_s\right]$$

since: $v = 1\,\frac{ft}{sec}$ $2\cdot b = 1\,\frac{ft}{sec}$

$$F_{SS} := K\cdot(\delta_s - 2\cdot a)$$

$(\delta_s - 2\cdot a) = 0.115\,in$

is the final deflection of spring

$F_{SS} = 1.15 \times 10^4\,lb$ as the graph shows !