**Problem 3 (30 points)**

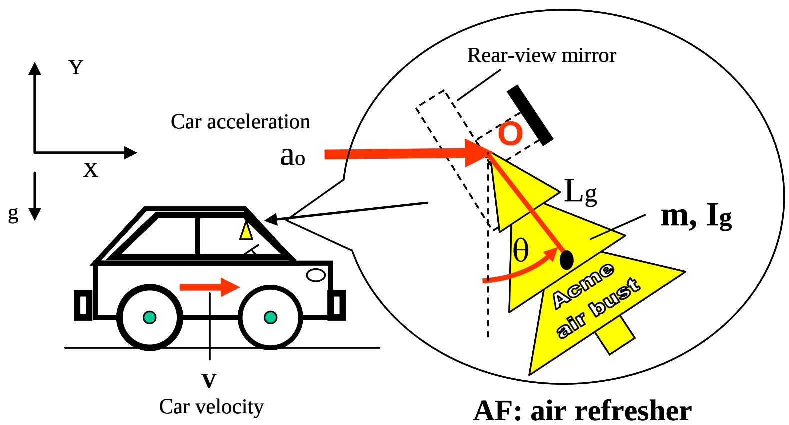
A car has been traveling for a while with speed $V_o$=100 km/hour. At time t=0 s, the car accelerates with $\mathbf{a_o}$ (m/s$^2$). The figure depicts a schematic view (not to scale) of an ACME air refresher (**AF**) hanging from the rear view mirror. The **AF** has mass properties equal to $m$ and $I_g=m\, r_k^2$, where $r_k$ is the radius of gyration; and $L_g$ is the distance from its center of mass to the support in the rear-view mirror. In the figure, $\theta$ denotes the angular displacement of the **AF** with respect to the vertical plane. Note that at time t=0 s, the **AF** is at rest, i.e. $\theta = 0, \dot{\theta} = 0$. **Tasks**

a) Derive the equation of motion for the **AF**. Express this equation in the form $\ddot{\theta} = f(L_g, r_k, m, g, a_o, \theta)$ [16]

b) Find an expression for the angular speed $\omega = \dot{\theta}$ [8]

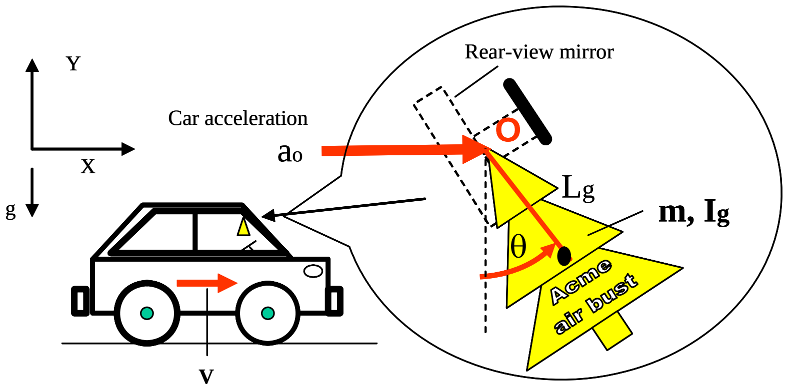
c) Find the maximum angular displacement $\theta_{max}$ if the car **decelerates** at 20 (km/hour)/s. [6]

Note: Energy is not conserved. Do not use PCME to derive the EOM




**Rear-view mirror pendulum** (not to scale)

# P3 - Motion with support (dec)acceleration




**AF: air refresher**

$I_g = m \cdot r_k^2$ mass moment of inertia about cg

$r_k = \text{radius\_of\_gyration}$

$L_g = \text{distance\_from\_cg\_to\_pivot\_O}$

$\vec{a_o} = a_o \cdot \vec{k}$ $a_o$ : acceleration of pivot

If $a_o < 0$ (deceleration)

$\vec{bog} = L_g \cdot \left( \sin(\theta) \cdot \vec{I} - \cos(\theta) \cdot \vec{J} \right)$

vector from pivot to cg

$I_o = I_g + m L_g^2$ by parallel axis theorem

## (a) Find equation of motion

**Use Moment equation for pivot (support) moving with known acceleration**

$$\sum M_o = I_o \cdot \frac{d^2}{dt^2}\theta + m \cdot \left( \vec{bog} \times \vec{a_o} \right) \quad [1]$$

$$\vec{bog} \times \vec{a_o} = a_o \cdot L_g \cdot \cos(\theta) \cdot \vec{K} \quad [2]$$

since $\vec{J} \cdot \vec{I} = -\vec{K}$

The only moment about pivot O is due to the weight action, i.e. $M_o = -m \cdot g \cdot L_g \cdot \sin(\theta)$ [3]

Substitution of eqs. [2,3] into [1] gives the EOM:

$$I_o \cdot \frac{d^2}{dt^2}\theta + m \cdot a_o \cdot L_g \cdot \cos(\theta) + m \cdot g \cdot L_g \cdot \sin(\theta) = 0 \quad [4]$$

Eq. [4] is identical to simple pendulum eqn if $a_o=0$ (pivot not moving with acceleration)

define: $L_o^2 = L_g^2 + r_k^2$ [5] where Lo is the "radius of gyration" about O

Thus, the EOM becomes

$$\frac{d^2}{dt^2}\theta = \frac{L_g}{L_o^2} \cdot \left( -g \cdot \sin(\theta) - a_o \cdot \cos(\theta) \right) \quad [6]$$

where $L_o^2 = L_g^2 + r_k^2$

## (b) Find angular velocity

Use the integral substitution $\frac{d^2}{dt^2}\theta = \frac{d}{d\theta}\left(\frac{\omega^2}{2}\right)$ where $\omega = \frac{d}{dt}\theta$ ; and write eq. [6] as

$$\frac{d}{d\theta}\left(\frac{\omega^2}{2}\right) = \frac{L_g}{L_o^2} \cdot \left( -g \cdot \sin(\theta) - a_o \cdot \cos(\theta) \right) \quad [7]$$

The initial conditions are: **motion starts from rest, i.e. at θ=0, ω=0**

Integrate Eq. [7] to obtain:

$$\frac{\omega^2}{2} = \frac{L_g}{L_o^2} \cdot \left[ g \cdot (\cos(\theta) - 1) - a_o \cdot \sin(\theta) \right]$$

Let $a_o = z \cdot g$

$$\frac{\omega^2}{2} = \frac{L_g \cdot g}{L_o^2} \cdot \left( \cos(\theta) - 1 - z \cdot \sin(\theta) \right) \quad [8]$$

## (c) Find maximum angular displacement

for maximum angular displacement $\beta=\theta_{max}$, set $\omega$=0 in eq. [8] to obtain: $(\cos(\beta) - 1) - z \cdot \sin(\beta) = 0$

$$\cos(\beta) - z \cdot \sin(\beta) = 1 \quad [9]$$

solve this equation with a calculator or:

square both sides of equation to obtain $\cos(\beta)^2 - 2 \cdot z \cdot \cos(\beta) \cdot \sin(\beta) + z^2 \cdot \sin(\beta)^2 = 1$

Using the identity $\cos(\beta)^2 = 1 - \sin(\beta)^2$ above, reduces Eq. above to $(z^2 - 1) \cdot \sin(\beta)^2 - 2 \cdot z \cdot \cos(\beta) \cdot \sin(\beta) = 0$

or $\sin(\beta) \cdot \left[(z^2 - 1) \cdot \sin(\beta) - 2 \cdot z \cdot \cos(\beta)\right] = 0$ [10]

This equation has two roots, $\sin(\beta)=0$, i.e. $\beta=0$ which is a trivial solution; and $(z^2 - 1) \cdot \sin(\beta) - 2 \cdot z \cdot \cos(\beta) = 0$

from which $\tan(\beta) = \left(\frac{2 \cdot z}{z^2 - 1}\right)$ where $z = \frac{a_o}{g}$

**For the problem data given,**

$kmh := \frac{1000 \cdot m}{3600 \cdot s}$

car initial speed $vel := 100 \cdot kmh$

deceleration $a_o := -20 \cdot \frac{kmh}{s}$ $a_o = -5.556\ m\,s^{-2}$

$z := \frac{a_o}{g}$ $z = -0.567$

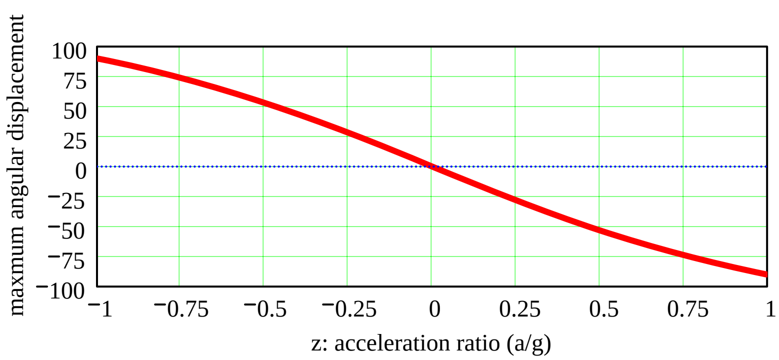
$\beta := \operatorname{atan}\left(\frac{2 \cdot z}{z^2 - 1}\right) \cdot \frac{180}{\pi}$

Maximum angle $\theta$max

$\beta = 59.064$ degrees

**NOT for exam**: Graph the max-angle for a number of acceleration ratios (respect to g)

$$\theta_{max}(z) := \operatorname{atan}\left(\frac{2 \cdot z}{z^2 - 1}\right) \cdot \frac{180}{\pi}$$




when z=1, $g = 9.807\ m\,s^{-2}$

$g = 35.304 \frac{kph}{s}$ (acceleration rate)

$\theta_{max}(.999) = -89.943$ 90 degree