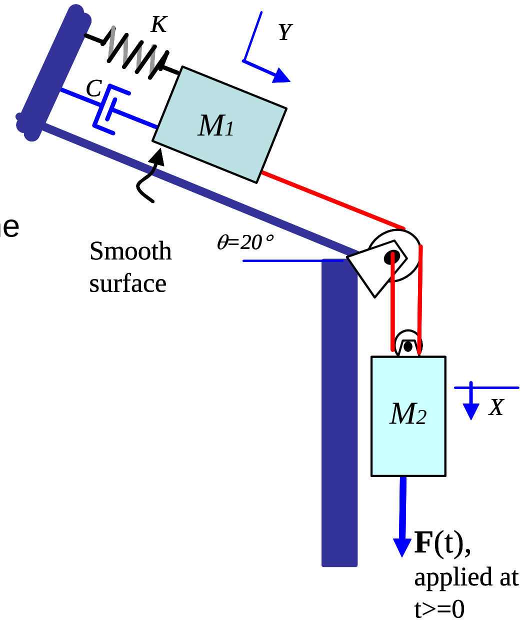
DERIVATION OF EQUATION OF MOTION

Select a coordinate system for motions
FROM static equilibrium position,
X(t) is the coordinate for motion of mass M2 along the vertical direction, + downwards,
Y(t) is the coordinate for motion of mass M1 along the inclined plane, + downwards and to the right



$W_1 := 5000 \cdot lb \qquad K := 10^5 \cdot \frac{lb}{in}$

$W_2 := 1000 \cdot lb$

$\theta := 20 \cdot \frac{\pi}{180}$ angle of inclined plane

$C := 150 \cdot lb \cdot \frac{sec}{in} \qquad M_1 := \frac{W_1}{g}$

$M_2 := \frac{W_2}{g}$

Static equilibrium position defines origin of coordinates *X*, *Y* describing the motion of blocks 2 and 1, respectively.

(a) kinematic constraint - inextensible cable

The cable length is constant, thus

$l_c = l_c + 2 \cdot X - Y$ **and the kinematic constraint follows as** $Y = 2 \cdot X$ **(1)**

(b) Static deflection of spring

By definition of SEP (Static equilibrium position), i.e. system is NOT moving and without any external forces applied into the system:

cable tension $2 \cdot T = W_2$ **must hold weight 2**

Static spring force $F_s = T + W_1 \cdot \sin(\theta)$ **must hold a fraction of weight 1+ cable tension**

hence

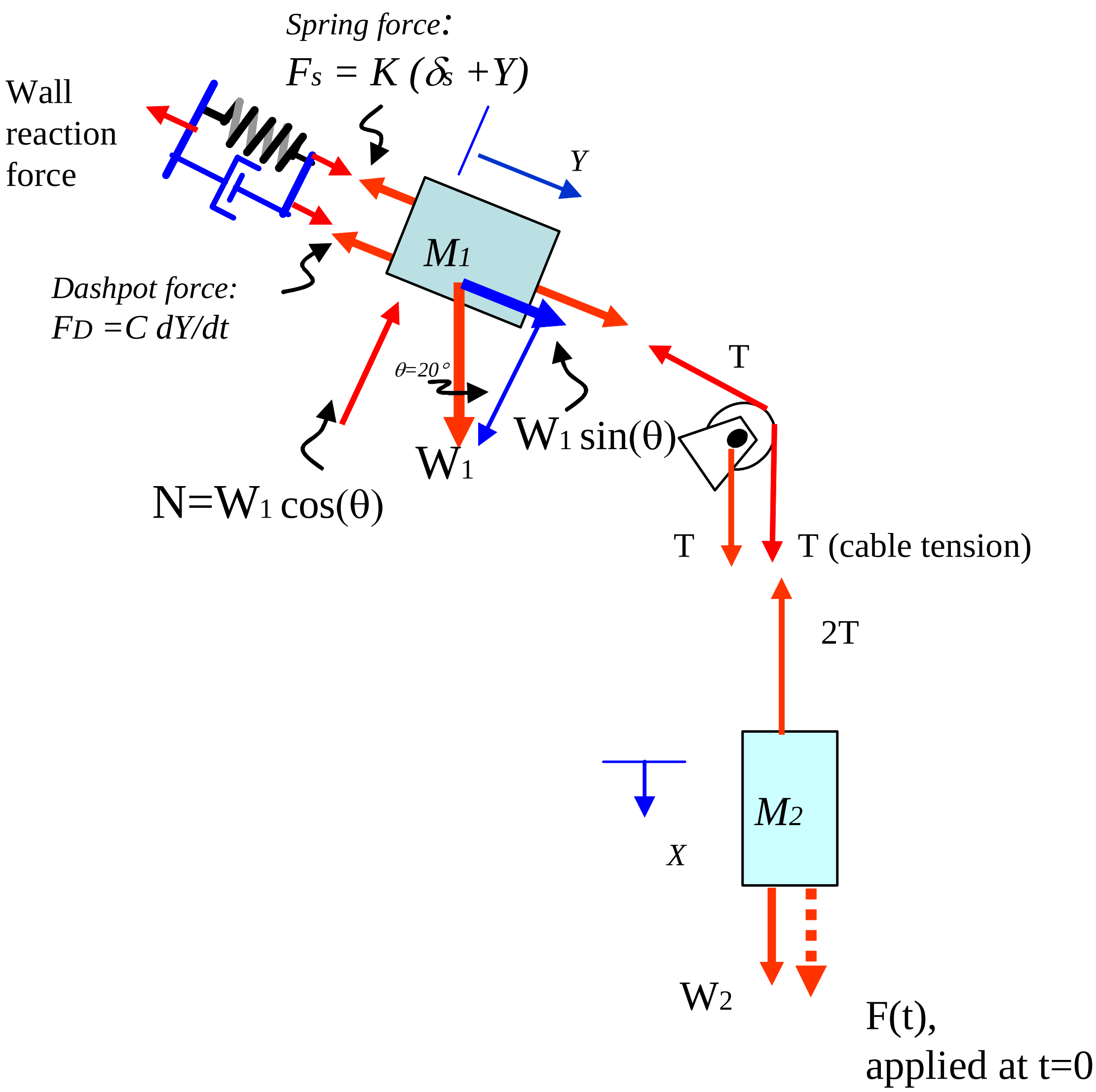
$$F_s = \frac{W_2}{2} + W_1 \cdot \sin(\theta) = K \cdot \delta_s \qquad \textbf{(2)}$$

δs is the static deflection of the spring needed to hold the system (w/o motion)

$$\delta_s := \frac{\frac{W_2}{2} + W_1 \cdot \sin(\theta)}{K} \qquad \textbf{(3)} \qquad \delta_s = 0.022\ in$$

Free Body Diagrams

Assumed state of motion to draw FBDs : $X>0$, $Y>0$



(c) for t>0, external force F(t) is applied to block 2.

Assume state of motion with X>0, Y>0 and draw free body diagrams:

From the FBD diagrams, apply Newton's 2nd law to obtain:

BLOCK 2

$$M_2 \cdot \frac{d^2}{dt^2} X = F(t) - 2 \cdot T + W_2 \qquad \textbf{(4)}$$

BLOCK 1

$$M_1 \cdot \frac{d^2}{dt^2} Y = W_1 \cdot \sin(\theta) + T - F_S - F_D \qquad \textbf{(5)}$$

where

$$F_{Damper} = C \cdot \frac{d}{dt} Y$$ is a viscous drag force **(6)**

$$F_S = (K \cdot Y + K \cdot \delta_S)$$ is the spring elastic force

(d) Derive single EOM for block motion

Note: EOM cannot contain internal forces (Tension for example). The tension is DETERMINED by the motion.

Substitute Eq. (6) into Eq. (5) and isolate the TENSION for substitution into Eq. (4)

$$M_1 \cdot \frac{d^2}{dt^2} Y = W_1 \cdot \sin(\theta) + T - \left(K \cdot Y + K \cdot \delta_s\right) - C \cdot \frac{d}{dt} Y$$

$$T = M_1 \cdot \frac{d^2}{dt^2} Y + \left(K \cdot Y + K \cdot \delta_s\right) + C \cdot \frac{d}{dt} Y - W_1 \cdot \sin(\theta) \qquad \textbf{(7)}$$

Multiply Eq. (7) times 2 and substitute into Eq.(4) to obtain:

$$M_2 \cdot \frac{d^2}{dt^2} X = F(t) + W_2 - 2 \cdot \left[M_1 \cdot \frac{d^2}{dt^2} Y + \left(K \cdot Y + K \cdot \delta_s\right) + C \cdot \frac{d}{dt} Y - W_1 \cdot \sin(\theta)\right] \qquad \textbf{(*)}$$

But recall that, from SEP condition: $\frac{W_2}{2} + W_1 \cdot \sin(\theta) = K \cdot \delta_s$

Hence, eq. (*) simplifies to

$$M_2 \cdot \frac{d^2}{dt^2} X = F(t) - 2 \cdot \left(M_1 \cdot \frac{d^2}{dt^2} Y + K \cdot Y + C \cdot \frac{d}{dt} Y\right)$$

and using the constraint **Y=2X**

$$\left(M_2 + 4 \cdot M_1\right) \cdot \frac{d^2}{dt^2} X + 4 \cdot K \cdot X + 4 \cdot C \cdot \frac{d}{dt} X = F(t) \qquad \textbf{(8)}$$

final EOM

On the other hand, if using Y(t) as the independent variable, the final EOM is

$$\left(\frac{M_2}{4} + M_1\right) \cdot \frac{d^2}{dt^2} Y + K \cdot Y + C \cdot \frac{d}{dt} Y = \frac{F(t)}{2} \qquad \textbf{(8b)}$$

(e) Calculate natural frequency and viscous damping ratio:

Use eq. (8) to continue wthe problem. Define equivalent physical parameters

$M_{eq} := M_2 + 4 \cdot M_1$ $\qquad K_{eq} := 4 \cdot K$ $\qquad C_{eq} := 4 \cdot C$

EOM: **(9)**

$$M_{eq} \cdot \frac{d^2}{dt^2} X + K_{eq} \cdot X + C_{eq} \cdot \frac{d}{dt} X = F(t)$$

$M_{eq} \cdot g = 2.1 \times 10^4 \, lb$

$K_{eq} = 4.8 \times 10^6 \, \frac{lb}{ft}$

$M_{eq} = 652.7 \, \frac{lb \, sec^2}{ft}$

Define:

$$\omega_n := \left(\frac{K_{eq}}{M_{eq}} \right)^{.5}$$

$\omega_n = 85.756 \, \frac{1}{sec}$

$f_n := \frac{\omega_n}{2 \cdot \pi}$

$$\zeta := \frac{C_{eq}}{2 \cdot \left(K_{eq} \cdot M_{eq} \right)^{.5}}$$

$f_n = 13.648 \, Hz$

$\zeta = 0.064$

$$\omega_d := \omega_n \cdot \left(1 - \zeta^2 \right)^{0.5}$$

$\omega_d = 85.578 \, \frac{rad}{sec}$ $\qquad f_d := \frac{\omega_d}{2 \cdot \pi}$ $\qquad T_d := \frac{1}{f_d}$

The damping ratio is not too large - motion will be oscillatory and will be damped out!

The damped natural frequency and period of motion are:

$f_d = 13.62 \, Hz$ $\qquad T_d = 0.073 \, sec$

<u>IMPORTANT NOTEs</u>:

Selecting the coordinate systems, X & Y, from the STATIC EQUILIBRIUM POSITION, i.e. from the assembled condition makes SENSE to your lecturer. It does NOT make sense (to him) choosing the origin of the coordinate system when the support cable IS NOT stretched nor compressed, i.e. Fs=K Y

The coordinate system is a RULER FIXED in space, allowing us (modelers) to FOLLOW (predict) the motion.

The coordinate system is NOT ATTACHED to the "center of mass" of a block. In this case, Newton's law will NOT be applicable.

A coordinate system(s) is NOT chosen because it gives PLUS or MINUS motion, whatwever this means.

Derive EOM using PCME

velocities

Kinetic Energy

$$T = \frac{1}{2}\cdot M_2\cdot V_2{}^2 + \frac{1}{2}\cdot M_1\cdot V_1{}^2$$

but $V_1 = V_2\cdot 2$

$$T = \frac{1}{2}\cdot(M_2 + 4\cdot M_1)\cdot\left(\frac{d}{dt}X\right)^2$$

since **Y=2X**

$$M_{eq} = M_2 + 4\cdot M_1$$

Potential energy

$$V = \frac{1}{2}\cdot K\left(Y + \delta_s\right)^2 - W_2\cdot X - W_1\cdot\sin(\theta)\cdot Y$$

$$V = \frac{1}{2}\cdot K\left(2\cdot X + \delta_s\right)^2 - W_2\cdot X - W_1\cdot\sin(\theta)\cdot 2\cdot X$$

Dissipated power

$$P_D = C\cdot\left(\frac{d}{dt}Y\right)^2 = 4C\cdot\left(\frac{d}{dt}X\right)^2$$

External power

$$P_E = F\cdot\left(\frac{d}{dt}X\right)$$

Apply PCME: $\frac{d}{dt}(T + V) = P_E - P_D$

$$\left[\frac{d^2}{dt^2}X\cdot(M_2 + 4\cdot M_1) + \left[K\cdot\left(2\cdot X + \delta_s\right)\cdot 2\right] - W_2 - 2\cdot W_1\cdot\sin(\theta)\right] = \left(F - 4C\cdot\frac{d}{dt}X\right)$$

But since $\frac{W_2}{2} + W_1\cdot\sin(\theta) = K\cdot\delta_s$ and $\left(\frac{d}{dt}X\right) \neq 0$ for all times

$$(M_2 + 4\cdot M_1)\cdot\frac{d^2}{dt^2}X + 4\cdot K\cdot X + 4\cdot C\cdot\frac{d}{dt}X = F(t)$$

(f) Example: Applied force is constant, i.e. STEP force

Let $F_o := 10000 \cdot lb$

Hence: $F(t) := F_o$ **(9)**

$$M_{eq} \cdot \frac{d^2}{dt^2} X + C_{eq} \cdot \frac{d}{dt} X + K_{eq} \cdot X = F(t) = F_o \quad (8)$$

What is steady-state motion?

Since F(t) is a constant, the particular solution of Eq (10) is:

$$X_p := \frac{F_o}{K_{eq}} \quad (10)$$

i.e blocks reach a new SEP

$$\frac{d}{dt} X_p = 0 \qquad \frac{d^2}{dt^2} X_p = 0$$

(g) find the full transient response - dynamic motion of block

Complete solution:

$$X(t) = X_H + X_p$$

$$X(t) = e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(C_1 \cdot \cos\left(\omega_d \cdot t\right) + C_2 \cdot \sin\left(\omega_d \cdot t\right)\right) + X_P \quad (11)$$

$$\frac{d}{dt} X = e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(D_1 \cdot \cos\left(\omega_d \cdot t\right) + D_2 \cdot \sin\left(\omega_d \cdot t\right)\right) \quad (12)$$

$$\text{Where } D_1 = -\zeta \cdot \omega_n \cdot C_1 + C_2 \cdot \omega_d$$

$$D_2 = -\zeta \cdot \omega_n \cdot C_2 - C_1 \cdot \omega_d \quad (13)$$

satisfy initial conditions at t=0:

$X_o := 0 \cdot ft$ $\quad V_o := 0 \cdot \frac{ft}{sec}$ **motion starts from rest**

from (11) and (12) at time t=0 sec

$$X_o = C_1 + X_P$$

$$V_o = D_1$$

$$X_p = 2.083 \times 10^{-3} ft$$

$$C_1 := X_o - X_p \qquad D_1 := V_o$$

$$C_2 := \frac{D_1 + \zeta \cdot \omega_n \cdot C_1}{\omega_d}$$

$$\zeta = 0.064$$

$$D_1 := -\zeta \cdot \omega_n \cdot C_1 + C_2 \cdot \omega_d \qquad D_2 := -\zeta \cdot \omega_n \cdot C_2 - C_1 \cdot \omega_d$$

and from (13)

$$C_1 = -0.025\,\text{in} \qquad C_2 = -1.611 \times 10^{-3}\,\text{in}$$

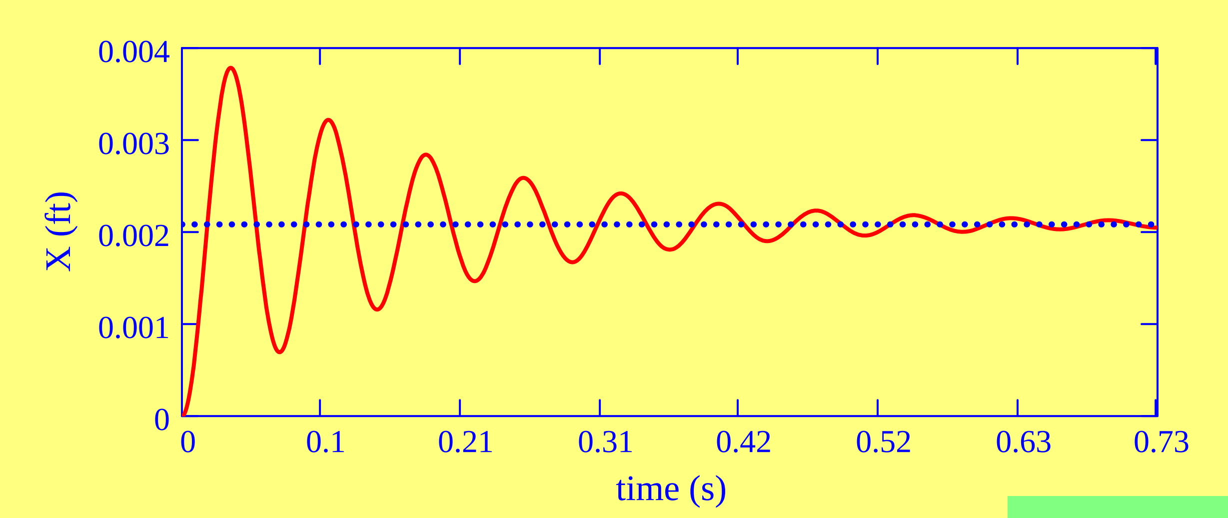
$$D_1 = 0\,\frac{\text{in}}{\text{sec}} \qquad D_2 = 2.148\,\frac{\text{in}}{\text{sec}}$$

Let's graph the response for time values up to 10 x damped period (my choice)

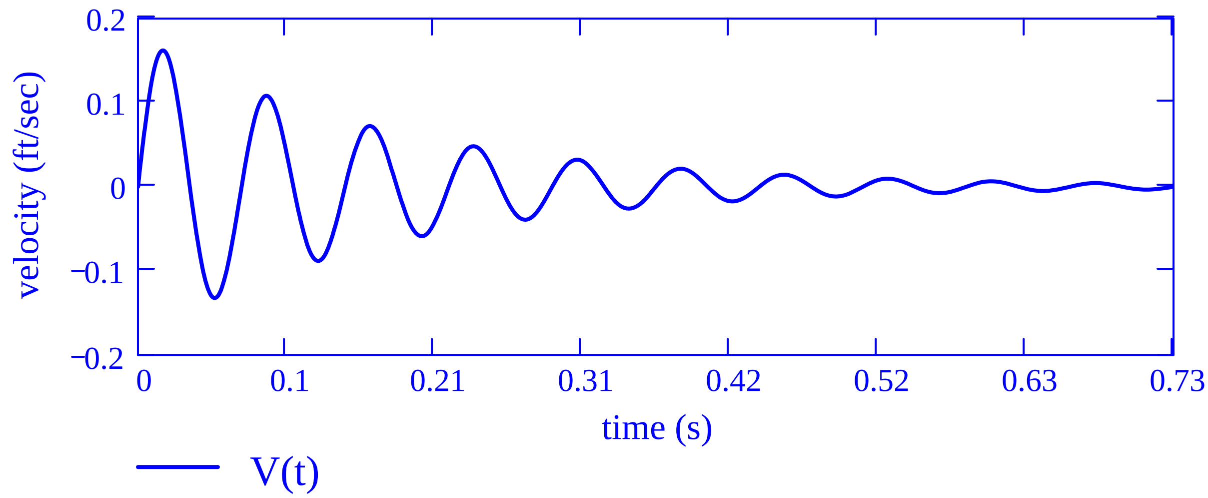
$$X(t) := e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(C_1 \cdot \cos\left(\omega_d \cdot t\right) + C_2 \cdot \sin\left(\omega_d \cdot t\right)\right) + X_p$$

$$V(t) := e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(D_1 \cdot \cos\left(\omega_d \cdot t\right) + D_2 \cdot \sin\left(\omega_d \cdot t\right)\right)$$

$$T_{max} := 10 \cdot T_d$$

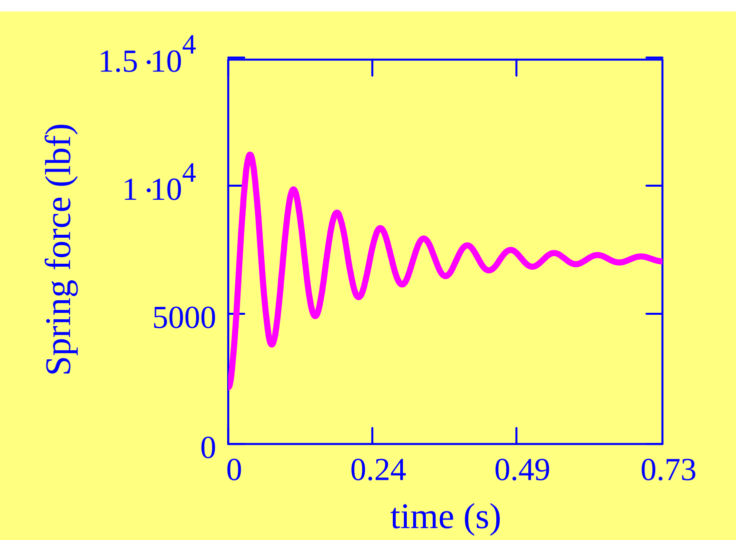


$X_p = 2.083 \times 10^{-3}\,\text{ft}$



Spring (cable) force (dynamic+static)

$F_s(t) := K \cdot \left(\delta_s + 2 \cdot X(t)\right)$ since Y=2 X



$K \cdot \delta_s = 2.21 \times 10^3\,\text{lb}$

$F_s(0 \cdot \text{sec}) = 2.21 \times 10^3\,\text{lb}$

At S-S

$F_s(1000 \cdot \text{sec}) = 7.21 \times 10^3\,\text{lb}$

$K \cdot \left(\delta_s + 2 \cdot X_p\right) = 7.21 \times 10^3\,\text{lb}$

Steady state value