Example: system response due to periodic function

Consider a 2nd order system described by the following EOM

L San Andres (c) 2008

$ORIGIN := 1$

$$M \cdot \frac{d^2}{dt^2} Y + C \cdot \frac{d}{dt} Y + K \cdot Y = K \cdot z(t)$$

where z (t) is an external periodic excitation function

Given the system parameters $M := 100 \cdot kg$ $K := 10^6 \cdot \frac{N}{m}$ $\zeta := 0.10$

calculate natural frequency and physical damping

$$\omega_n := \left(\frac{K}{M}\right)^{0.5} \qquad f_n := \frac{\omega_n}{2 \cdot \pi} \qquad f_n = 15.915\,Hz$$

$$C := 2 \cdot M \cdot \omega_n \cdot \zeta \qquad C = 2 \times 10^3\, s \frac{N}{m}$$

$$\omega_d := \omega_n \cdot \left(1 - \zeta^2\right)^{.5} \qquad f_d := \frac{\omega_d}{2 \cdot \pi} \qquad T_d := \frac{1}{f_d} \qquad T_d = 0.063\,s$$

damped natural period

Define periodic excitation function:

Example - square wave

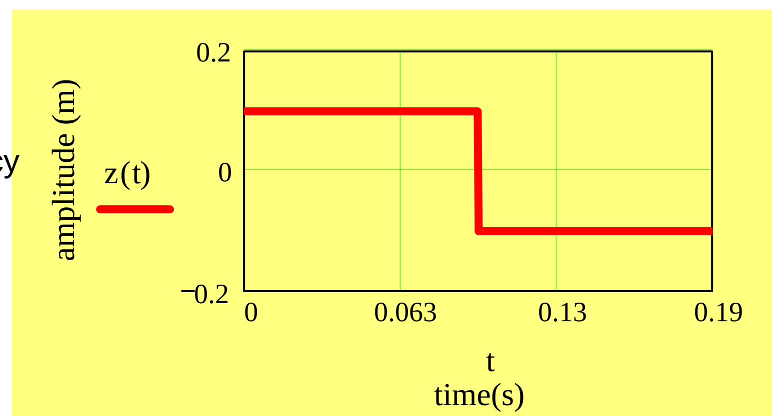
$$T := \frac{T_d}{.33333} \qquad z_o := 0.1 \cdot m$$

$$z(t) := \begin{array}{|l} amp \leftarrow z_o \;\; \text{if} \;\; t < \frac{T}{2} \\ amp \leftarrow -z_o \;\; \text{if} \;\; t > \frac{T}{2} \\ amp \end{array}$$

$$\Omega := \frac{2 \cdot \pi}{T}$$ fundamental frequency

$$\frac{\Omega}{\omega_d} = 0.333$$

$N_F := 7$ number of Fourier coefficients



Find Fourier Series coefficients for excitation z(t)

mean value $a_0 := \frac{1}{T} \cdot \int_0^T z(t)\, dt$ $a_0 = 0\,m$

$j := 1 .. N_F$

coefs of cos & sin

$$a_j := \frac{2}{T} \cdot \int_0^T z(t) \cdot \cos(j \cdot \Omega \cdot t)\, dt \qquad b_j := \frac{2}{T} \cdot \int_0^T z(t) \cdot \sin(j \cdot \Omega \cdot t)\, dt$$

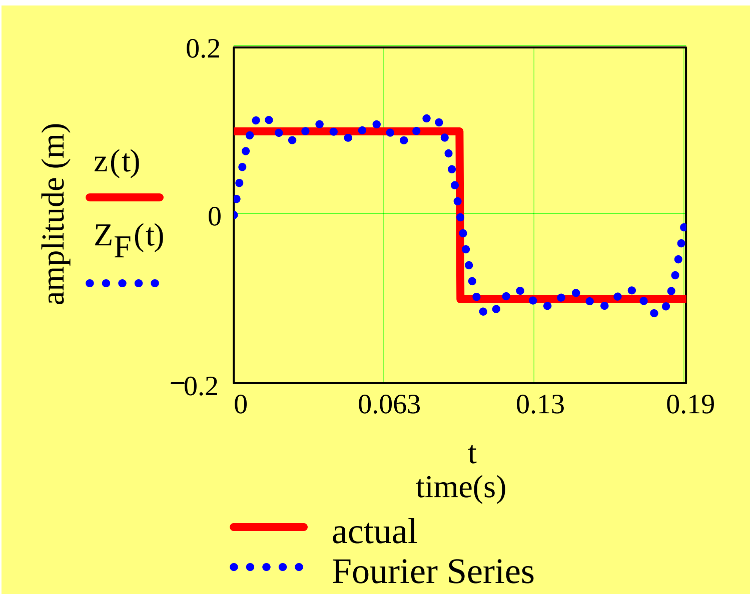
$a^T = (0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0)\, m$ $\qquad b^T = (0.127 \;\; 0 \;\; 0.042 \;\; 0 \;\; 0.025 \;\; 0 \;\; 0.018)\, m$

Build z(t) as a Fourier series

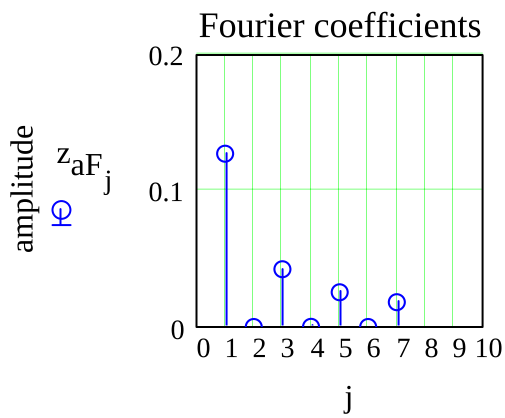
$$Z_F(t) := a_0 + \sum_{j=1}^{N_F} \left(a_j \cdot \cos(j \cdot \Omega \cdot t) + b_j \cdot \sin(j \cdot \Omega \cdot t)\right)$$

$$z_{aF_j} := \left[(a_j)^2 + (b_j)^2\right]^{.5}$$

Amplitude



$N_F = 7$



Find the forced response of the system, i.e, find Y(t)

SYSTEM RESPONSE is:

$$Y(t) := Y_o + \sum_{m=1}^{N_F} \left[\left(Yc_m \cdot \cos(m \cdot \Omega \cdot t) + Ys_m \cdot \sin(m \cdot \Omega \cdot t)\right)\right]$$

$$Y_o := a_0 \cdot \frac{K}{K}$$

$m := 1 .. N_F$

(a) set frequency ratio $\quad f_m := \frac{m \cdot \Omega}{\omega_n}$

(b) build denominator

$$den_m := \left[1 - (f_m)^2\right]^2 + (2\cdot\zeta\cdot f_m)^2$$

(c) build coefficient of cos()

$$Yc_m := \frac{K}{K}\cdot\frac{\left[a_m\cdot\left[1-(f_m)^2\right] - 2\cdot\zeta\cdot f_m\cdot b_m\right]}{den_m}$$

(d) build coefficient of sin()

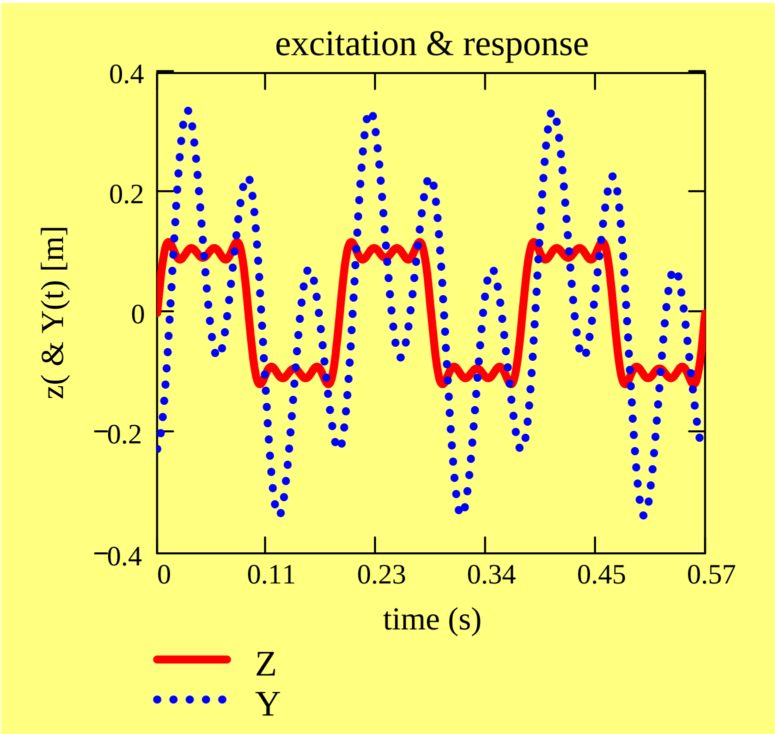
$$Ys_m := \frac{K}{K}\cdot\frac{\left[b_m\cdot\left[1-(f_m)^2\right] + 2\cdot\zeta\cdot f_m\cdot a_m\right]}{den_m}$$

(e) for graph of components

$$Y_{F_m} := \left[(Yc_m)^2 + (Ys_m)^2\right]^{.5}$$

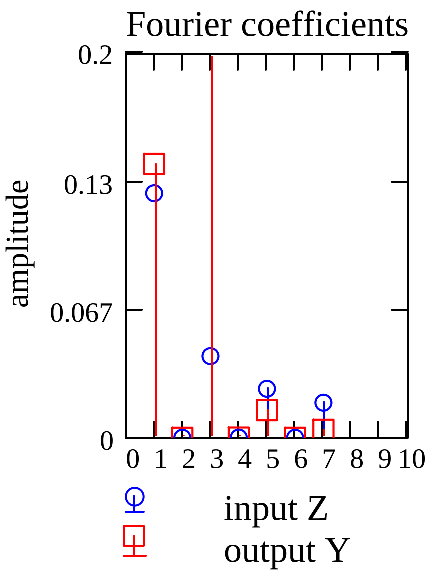
$$Y(t) := Y_o + \sum_{m=1}^{N_F}\left(Yc_m\cdot\cos(m\cdot\Omega\cdot t) + Ys_m\cdot\sin(m\cdot\Omega\cdot t)\right)$$

Now graph the response Y(t) and the excitation (Fourier) z(t):



$$\frac{T}{T_d} = 3$$

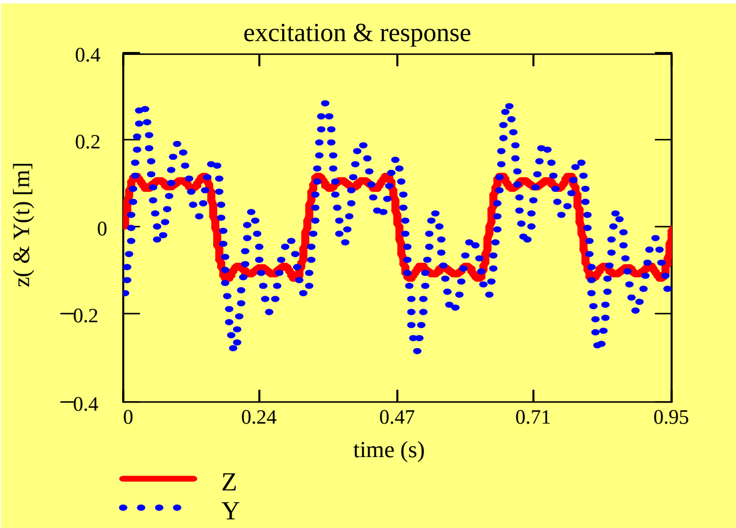
$$\frac{\Omega}{\omega_n} = 0.332$$



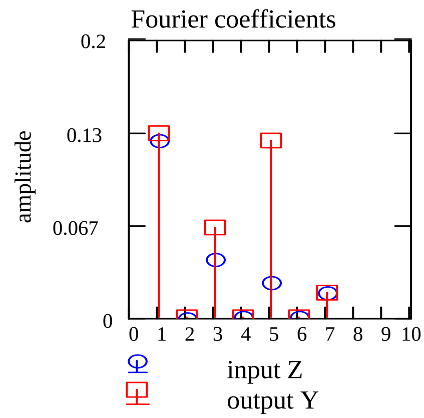
3 periods of fundamental excitation motion

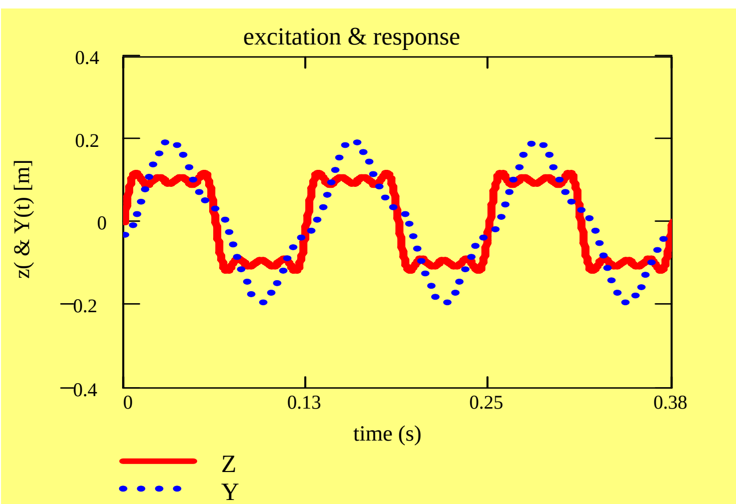
Note: obtain response for inputs with increasing frequencies (periods decrease)

==

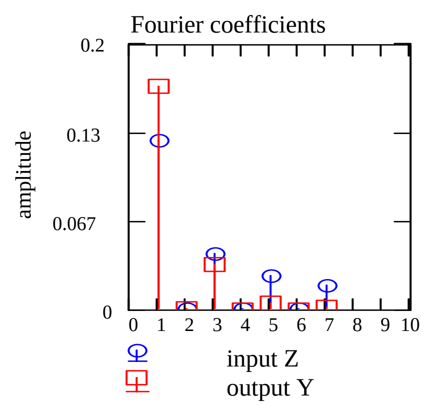


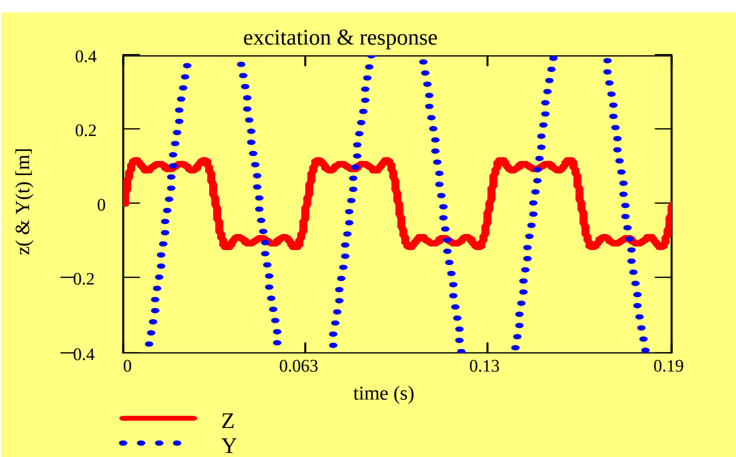
$$\frac{T}{T_d} = 5 \qquad \frac{\Omega}{\omega_n} = 0.199$$



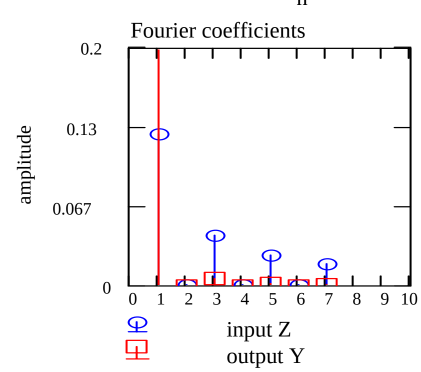


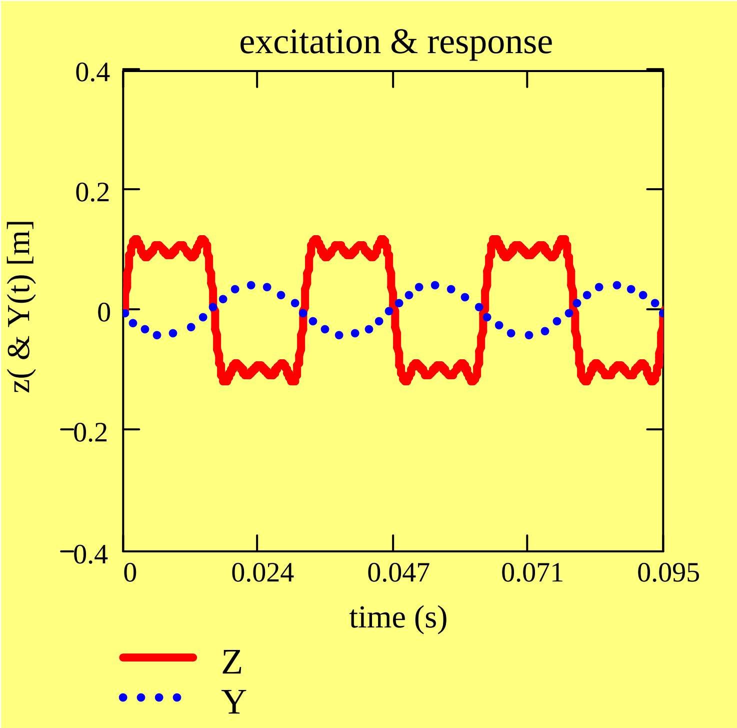
$$\frac{T}{T_d} = 2 \qquad \frac{\Omega}{\omega_n} = 0.497$$



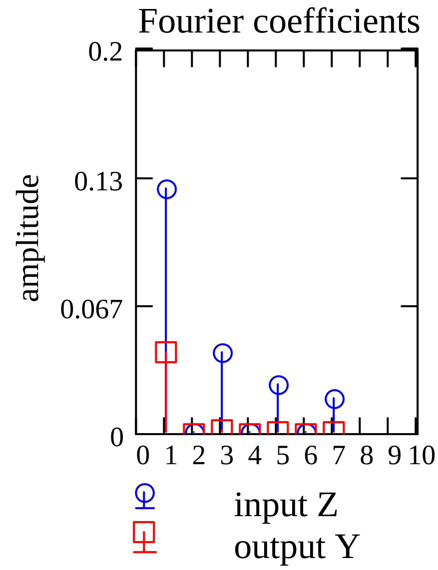


$$\frac{T}{T_d} = 1 \qquad \frac{\Omega}{\omega_n} = 0.995$$

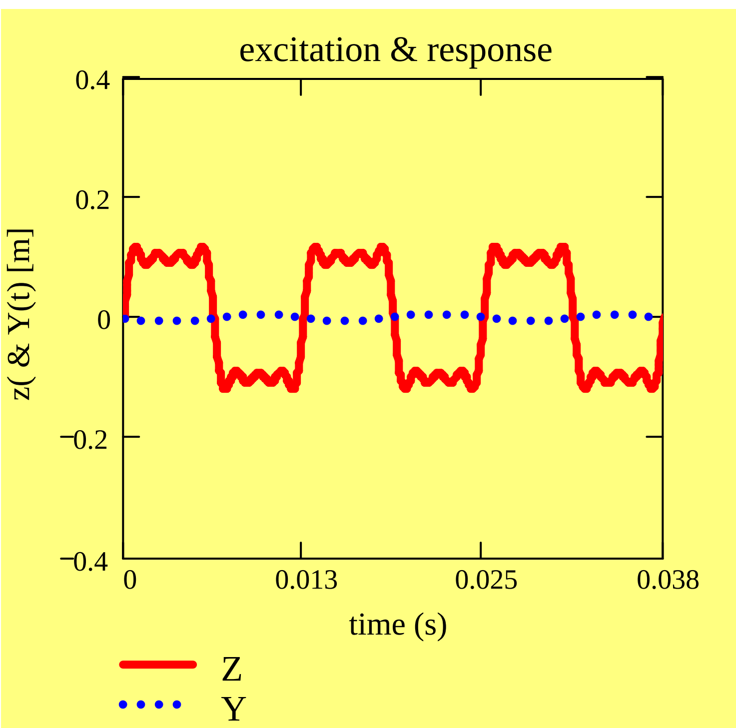




$$\frac{T}{T_d} = 0.5 \qquad \frac{\Omega}{\omega_n} = 1.99$$



== fastest Z (smallest period)



$$\frac{T}{T_d} = 0.2 \qquad \frac{\Omega}{\omega_n} = 4.975$$