MDOF: PERIODIC FORCED RESPONSE

$ORIGIN := 1$

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ME363 FALL 2010

Consider the n-DOF system governed by

$$M\cdot\frac{d^2}{dt^2}X + C\cdot\frac{d}{dt}X + K\cdot X = F(t) = F_o\cdot\cos(\Omega\cdot t) \qquad (1)$$

where X is a vector of n-displacements, F is a vector of periodic forces with frequency Ω. M,C,K are nxn matrices of inertia, damping and stiffness coefficients..

$X=\{ x1, x2...... xn\}^T$ $\qquad$ $F=\{ F1, F2...... Fn\}^T$

The force vector acts on the system over a long-long-long time. Hence, transient response effects have died and the periodic foced response of the system is

$$X = X_c\cdot\cos(\Omega\cdot t) + X_s\cdot\sin(\Omega\cdot t) \qquad (2)$$

where Xc and Xs are vectors of displacements associated to the cos & sine terms. Note that

$$\frac{d}{dt}X = -X_c\cdot\Omega\cdot\sin(\Omega\cdot t) + X_s\cdot\Omega\cdot\cos(\Omega\cdot t) \qquad (3)$$

$$\frac{d^2}{dt^2}X = -\Omega^2\cdot X = -X_c\cdot\Omega^2\cdot\cos(\Omega\cdot t) - X_s\cdot\Omega^2\cdot\sin(\Omega\cdot t)$$

Substitution of Eqs. (2) and (3) into Eq. (1) gives

$$\left(K - \Omega^2\cdot M\right)\cdot X_c\cdot\cos(\Omega\cdot t) + X_s\cdot\Omega\cdot C\cdot\cos(\Omega\cdot t) \qquad (4)$$

$$+ \; -X_c\cdot\Omega\cdot C\cdot\sin(\Omega\cdot t) + \left(K - \Omega^2\cdot M\right)\cdot X_s\cdot\sin(\Omega\cdot t) = F_o\cdot\cos(\Omega\cdot t)$$

The functions cos & sin are linearly independent. Hence, collecting similar terms we have

$$\left(K - \Omega^2\cdot M\right)\cdot X_c + C\cdot\Omega\cdot X_s = F_o \qquad (5)$$

$$-C\cdot\Omega\cdot X_c + \left[\left(K - \Omega^2\cdot M\right)\cdot X_s\right] = 0$$

Eqns. (5) are written in matrix form as

$$\begin{pmatrix} K - \Omega^2 \cdot M & \Omega \cdot C \\ -\Omega \cdot C & K - \Omega^2 \cdot M \end{pmatrix} \cdot \begin{pmatrix} X_c \\ X_s \end{pmatrix} = \begin{pmatrix} F_o \\ 0 \end{pmatrix} \qquad (6)$$

For n=2 (DOF)

This is a system of 4-equations with 4-unknowns which can be easily solved for any particular frequency (Ω)

Example:

let

$$M := \begin{pmatrix} 2.381 \times 10^3 & -113.398 \\ -113.398 & 566.99 \end{pmatrix} \cdot kg \qquad F_o := \begin{pmatrix} 10^4 \\ 0 \end{pmatrix} \cdot N$$

$$K := \begin{pmatrix} 1.226 \times 10^8 & -3.503 \times 10^7 \\ -3.503 \times 10^7 & 3.503 \times 10^7 \end{pmatrix} \cdot \frac{N}{m}$$

$$C := \begin{pmatrix} 10 & -2 \\ -3 & 3 \end{pmatrix} \cdot 4000 \cdot \frac{N \cdot s}{m}$$

Make matrix

$$G(\Omega) := \begin{pmatrix} K - \Omega^2 \cdot M & \Omega \cdot C \\ -\Omega \cdot C & K - \Omega^2 \cdot M \end{pmatrix}$$

$$G(\Omega) := \begin{pmatrix} K_{1,1} - \Omega^2 \cdot M_{1,1} & K_{1,2} - \Omega^2 \cdot M_{1,2} & \Omega \cdot C_{1,1} & \Omega \cdot C_{1,2} \\ K_{2,1} - \Omega^2 \cdot M_{2,1} & K_{2,2} - \Omega^2 \cdot M_{2,2} & \Omega \cdot C_{2,1} & \Omega \cdot C_{2,2} \\ -\Omega \cdot C_{1,1} & -\Omega \cdot C_{1,2} & K_{1,1} - \Omega^2 \cdot M_{1,1} & K_{1,2} - \Omega^2 \cdot M \\ -\Omega \cdot C_{2,1} & -\Omega \cdot C_{2,2} & K_{2,1} - \Omega^2 \cdot M_{2,1} & K_{2,2} - \Omega^2 \cdot M \end{pmatrix}$$

FInd solution over a range of frequencies

$N := 100$

$f_{min} := 10 \cdot Hz$ $\quad f_{max} := 80 \cdot Hz$

$\Omega_{min} := f_{min} \cdot 2 \cdot \pi$ $\quad \Omega_{max} := f_{max} \cdot 2 \cdot \pi$

$$\Delta\Omega := \frac{(\Omega_{max} - \Omega_{min})}{N - 1}$$

$j := 1 .. N$

$\Omega_j := \Omega_{min} + \Delta\Omega \cdot j$ $\quad freq_j := \frac{\Omega_j}{2 \cdot \pi}$

Solve:

$$\begin{pmatrix} X_{1c_j} \\ X_{2c_j} \\ X_{1s_j} \\ X_{2s_j} \end{pmatrix} := G(\Omega_j)^{-1} \cdot \begin{pmatrix} F_{o_1} \\ F_{o_2} \\ 0 \\ 0 \end{pmatrix}$$

force cos terms

force sin terms

for example, the response for DOF#1 is

$$X_1(t) = X_{1c} \cdot \cos(\Omega \cdot t) + X_{1s} \cdot \sin(\Omega \cdot t)$$

which can also be written as

$$X_1(t) = A_1 \cdot \cos(\Omega \cdot t + \phi_1)$$

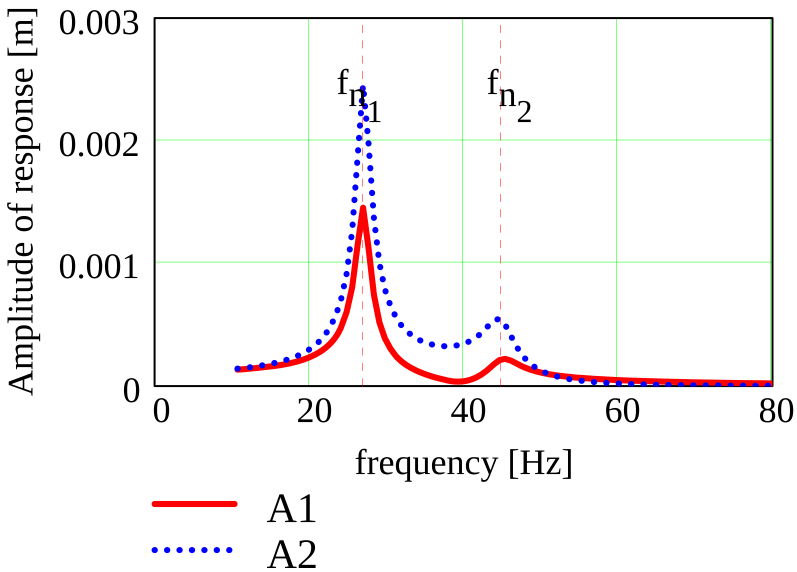
where A1 and $\phi 1$ denote the amplitude and phase lag of the response

Hence, define for each DOF:

$$A_{1_j} := \left[(X_{1c_j})^2 + (X_{1s_j})^2 \right]^{.5} \qquad \phi_{1_j} := -atan\left(\frac{X_{1s_j}}{X_{1c_j}}\right) \cdot \frac{180}{\pi}$$

$$A_{2_j} := \left[(X_{2c_j})^2 + (X_{2s_j})^2 \right]^{.5} \qquad \phi_{2_j} := -atan\left(\frac{X_{2s_j}}{X_{2c_j}}\right) \cdot \frac{180}{\pi}$$

PLOTS of the FRF: amplitude and phase angle of the system periodic response



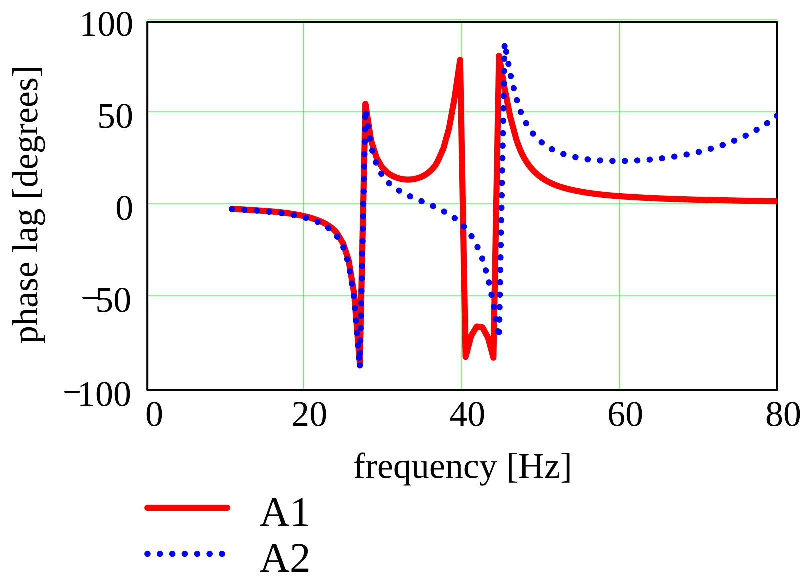
undamped natural freqs.

$$f_n = \begin{pmatrix} 27.019 \\ 44.903 \end{pmatrix} Hz$$

Questions from analysis:

IS RESPONSE SAFE?

AT WHICH FREQUENCIES SYSTEM OPERATION MUST BE AVOIDED?



Find UNDAMPED natural frequencies and mode shapes

let: $E := M^{-1} \cdot K$

$$\lambda := \text{eigenvals}(E)$$

$$f_n := \frac{\lambda^{.5}}{2 \cdot \pi}$$

$$f_n = \begin{pmatrix} 27.019 \\ 44.903 \end{pmatrix} \text{Hz}$$

$$a_1 := \text{eigenvec}(E, \lambda_1) \qquad a_2 := \text{eigenvec}(E, \lambda_2)$$

$$a_1 = \begin{pmatrix} 0.507 \\ 0.862 \end{pmatrix} \qquad a_2 = \begin{pmatrix} -0.362 \\ 0.932 \end{pmatrix}$$

$A := \text{augment}(a_1, a_2)$ MODAL matrix

MODAL matrices

$$A^T \cdot M \cdot A = \begin{pmatrix} 934.399 & 5.387 \times 10^{-14} \\ 5.465 \times 10^{-14} & 881.457 \end{pmatrix} \text{kg}$$

$$A^T \cdot K \cdot A = \begin{pmatrix} 2.693 \times 10^{7} & 6.361 \times 10^{-9} \\ 1.252 \times 10^{-8} & 7.016 \times 10^{7} \end{pmatrix} \frac{\text{kg}}{\text{s}^2}$$

$$A^T \cdot C \cdot A = \begin{pmatrix} 1.046 \times 10^{4} & 2.258 \times 10^{3} \\ -881.491 & 2.242 \times 10^{4} \end{pmatrix} \frac{\text{kg}}{\text{s}}$$

Modal damping is NOT diagonal