MEEN 617 Handout #12 The FEM in Vibrations

A brief introduction to the finite element method for modeling of mechanical structures

The **finite element method** (**FEM**) is a piecewise application of a **variational method.**

Here I provide you with a fundamental introduction to the FE method congruent with prior analysis of mechanical systems and the assumed modes method.

For a complete and most lucid introduction to the FEM please read Reddy, J.N., "**Introduction to the Finite Element Method**," John Wiley Pubs.

The beauty of the FEM method lies on its simplicity since its formulation is independent of the actual response of the system. That is, **little knowledge about an expected answer is required a-priori.**

Let's review the **Assumed Modes Method.**

In any mechanical system, the **Hamiltonian**

$$\delta\left\{\int_{t_1}^{t_2}\left(T-V+W_{ext}\right)dt\right\}=0 \tag{1}$$

is the fundamental principle of mechanics from which the laws of motion are derived, i.e., Newton's Laws and/or Lagrangian Mechanics.

In general, the kinetic energy (T) and the strain energy (V) of a mechanical system are functions of the displacement vector ( ) and its time derivatives, i.e.,

$$T = T\left(\vec{v}, \dot{\vec{v}}, \text{material properties}\right)$$
$$V = V\left(\vec{v}, \dot{\vec{v}}, \text{material properties}\right) \tag{2}$$

where (·) = d/dt and $\vec{\mathrm{v}} = \left(v_x \vec{\mathrm{i}} + v_y \vec{\mathrm{j}} + v_z \vec{\mathrm{k}}\right)$ is the vector of displacements of a material point in the domain of interest $\Omega = \Omega(x_i)$

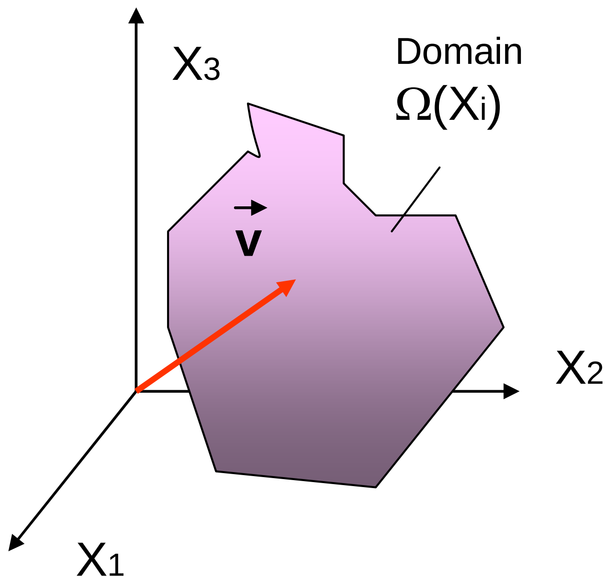


Fig 1. Domain for analysis

In the **assumed modes method** an approximation to the displacement function in a continuous system is expressed as

$$\vec{\mathbf{v}} = \sum_{i=1}^{n} \psi_{i(x_j)} v_{i_{(t)}} \tag{3}$$

where **each** $\psi_{i(\Omega)}$ describes a deflected shape over the entire domain. Eq. (3) is a linear combination of the **basis functions set** $\{\psi_{i(\Omega)}\}$

As shown in past lectures, substitution of Eq. (3) into Eq. (1) leads to an *N*-DOF mathematical model of the mechanical system, whose equations of motion are:

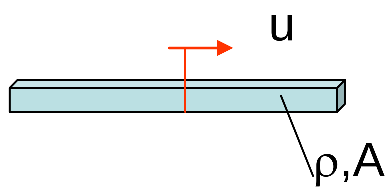
$$\mathbf{M}\,\ddot{\mathbf{v}} + \mathbf{K}\,\mathbf{v} = \mathbf{Q} \qquad (4)$$

where **M = M**^{T} and **K = K**^{T} are the *N* x *N* matrices of mass and stiffness coefficients, respectively, and **Q** is the vector of generalized forces.

The coefficients in **M** and **K** are determined from relations of the shape functions and its derivatives. **For example:**

For an elastic **bar subjected to axial motions**:

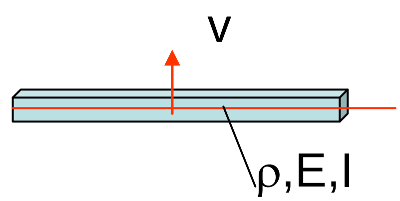
$$M_{ij} = M_{ji} = \int_0^L \rho\, A\, \psi_i \psi_j \, dx,$$



$$K_{ij} = K_{ji} = \int_0^L E\, A \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} dx \qquad (5.a)$$

while, for an elastic **beam under transverse or lateral deformations**,

$$M_{ij} = M_{ji} = \int_0^L \rho\, A\, \psi_i \psi_j \, dx,$$



$$K_{ij} = K_{ji} = \int_0^L E\, I \frac{d^2\psi_i}{dx^2}\frac{d^2\psi_j}{dx^2} dx \qquad (5.b)$$

Above, the **set of shape functions** $\{\psi_i\}_{\,i=1,2,..N}$ must satisfy be an **admissible set** (which means):

* $\{\psi_i\}$ must be a linearly independent set,
* satisfy the essential boundary conditions,
* be sufficiently differentiable as required by the strain energy function.

However, there are a number of problems associated with these requirements:

a) a complex system geometry requires of complex shape functions, i.e., a difficult task for the inexperienced user;
b) $\{\psi_i\}$ are defined over the entire domain (Ω) and thus, they lead to a highly coupled system of equations;
c) $\{\psi_i\}$ are related to a particular problem; and consequently, not general.

The **FEM** overcomes these difficulties and provides a sound basis for the analysis of vibrations of mechanical systems.

The **FEM** can be envisioned in the present context as an application of the assumed modes method wherein the shape functions $\{\psi_i\}$ represent deflection over just a portion (finite element) of the structure, with the elements being assembled to form the structural system.

A FE model for axial deformations of an elastic bar

Figure 2 shows an elastic bar (one fixed end) subjected to axial loads $(P, f)_{(t)}$ that cause axial displacements or elastic deformations $u_{(x,t)}$.

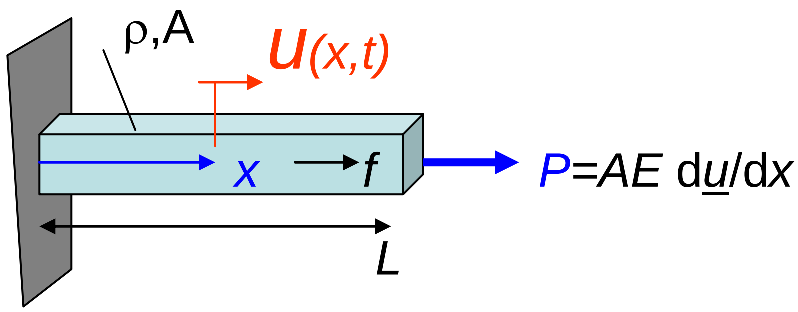


Fig 2. Elastic bar under axial displacements

The first step in the FEM is to divide the domain Ω into a series of finite elements $\{\Omega^e\}$ and then constructing a finite element mesh, as shown in Fig. 3.

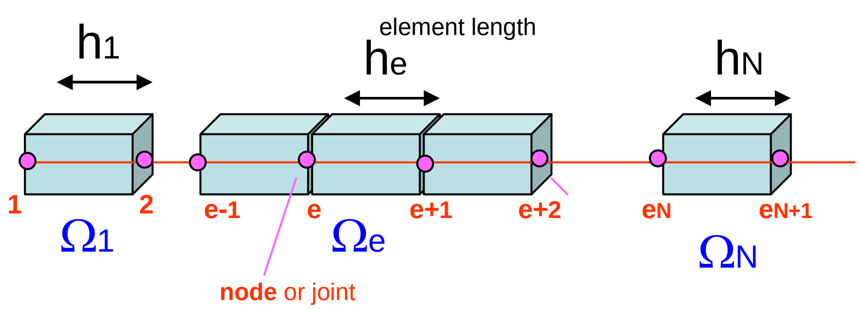


Fig 3. Discretization of bar into *N* finite elements

Derivation of EOM for one finite element

A typical element $\Omega^e = (x_A, x_B)$ is isolated from the mesh.

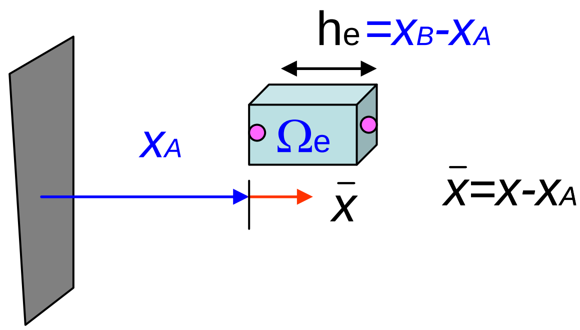


Figure 4 depicts a free body diagram of the element, where u_1^e and u_2^e are the (nodal) axial displacements at the tips of the element, and P_1^e and P_2^e are the (node) axial forces arising from the reactions with the neighboring elements. Note f represents a distributed force/unit length acting on the bar.

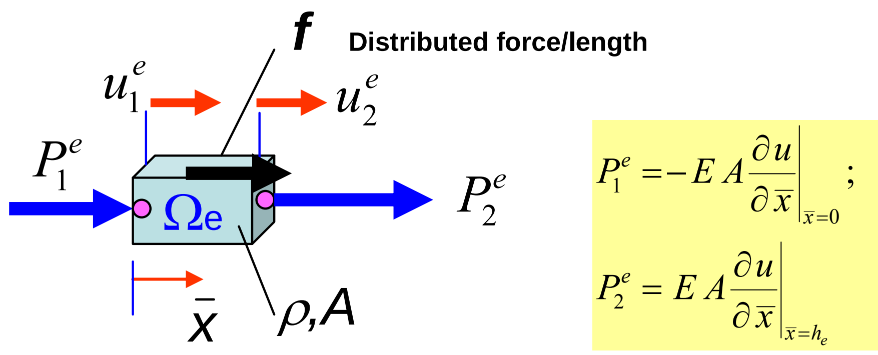


Fig 4. Free body diagram for a finite element in axial bar

The nodal axial forces P are

$$P_1^e = -E A \left.\frac{\partial u}{\partial \bar{x}}\right|_{\bar{x}=0} \; ; \; P_2^e = E A \left.\frac{\partial u}{\partial \bar{x}}\right|_{\bar{x}=h_e}$$

The kinetic energy (T^e) and potential (strain) energy (V^e) for the finite element Ω^e are:

$$T^e = \frac{1}{2}\int_0^{h_e} \rho A \left(\frac{\partial u}{\partial t}\right)^2 d\bar{x}, \qquad V^e = \frac{1}{2}\int_0^{h_e} E A \left(\frac{\partial u}{\partial \bar{x}}\right)^2 d\bar{x} \tag{6}$$

and the virtual work from the external forces over the element is

$$\delta W^e_{ext} = \int_0^{h_e} f_{(\bar{x})} \delta u^e \, d\bar{x} + \delta u^e_{(x_A)} P_1^e + \delta u^e_{(x_B)} P_2^e \tag{7}$$

The Hamiltonian Principle applies over the whole system; and hence, it also holds over the element (Ω^e); thus

$$\delta\left\{\int_{t_1}^{t_2} \left(T^e - V^e + W^e_{ext}\right) dt\right\} = 0 \tag{8}$$

In $\Omega^e = (x_A, x_B)$, let the displacement u^e be represented as

$$u^e_{(\bar{x},t)} = \sum_{i=1}^{2} \psi^e_{i_{(\bar{x})}} u^e_{i\,(t)} \tag{9}$$

where $u_1^e = u^e_{(x_A,t)}$, $u_2^e = u^e_{(x_B,t)}$ are the nodal displacements and ψ_1^e and ψ_2^e are shape functions **admissible** to the problem. Note that one presumes to know $\left(u_1^e, u_2^e\right)$.

Substitution of Eq. (9) into Eq. (6) gives:

$$T^e = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} M_{ij}^e \dot{u}_i^e \dot{u}_j^e, \quad V^e = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} K_{ij}^e u_i^e u_j^e \qquad (11)$$

where the coefficients of the **element mass** and **element stiffness** matrices are:

$$M_{ij}^e = M_{ji}^e = \int_0^{h_e} \rho^e A^e \psi_i^e \psi_j^e \, d\bar{x},$$
$$K_{ij}^e = K_{ji}^e = \int_0^{h_e} E^e A^e \frac{d\psi_i^e}{d\bar{x}} \frac{d\psi_j^e}{d\bar{x}} \, d\bar{x} \qquad (12)$$

The shape functions in Eq. (9) $u_{(\bar{x},t)}^e = \sum_{i=1}^{2} \psi_{i_{(\bar{x})}}^e u_{i\,(t)}^e = \psi_1^e u_1^e + \psi_2^e u_2^e$ must satisfy:

At $x = x_A$ or $\bar{x} = 0$,

$$u_{(0,t)}^e = u_1^e = \psi_{1_{(\bar{x}=0)}}^e u_1^e + \psi_{2_{(\bar{x}=0)}}^e u_2^e,$$

$$\rightarrow \psi_{1_{(\bar{x}=0)}}^e = 1, \; \psi_{2_{(\bar{x}=0)}}^e = 0 \qquad (13a)$$

At $x = x_B$ or $\bar{x} = h_e$,

$$u_{(h_e,t)}^e = u_1^e = \psi_{1_{(\bar{x}=h_e)}}^e u_1^e + \psi_{2_{(\bar{x}=h_e)}}^e u_2^e$$
$$\rightarrow \psi_{1_{(\bar{x}=h_e)}}^e = 0, \; \psi_{2_{(\bar{x}=h_e)}}^e = 1 \qquad (13b)$$

Note that from Eq. (12), the shape functions need to be at least once differentiable over the element Ω^e, i.e., $\left\{\psi_i^e\right\}_{i=1,..n} \in C^1_{\left(\Omega^e\right)}$.

Note $n > 2$ in Eq. (9) is also a choice. However, higher order elements are not discussed here for brevity.

Work now with the **virtual work** performed by the external forces. Substitution of Eq. (9) into Eq. (7) leads to:

$$\delta W_{ext}^e = \left(\int_0^{h_e} f_{(\bar{x})} \psi_i^e \, d\bar{x} \right) \delta u_i^e + \delta u_1^e \psi_{1_{(0)}}^e P_1^e + \delta u_2^e \psi_{2_{(h_e)}}^e P_2^e$$

$$\rightarrow \delta W_{ext}^e = F_i^e \delta u_i^e + \delta u_1^e P_1^e + \delta u_2^e P_2^e \qquad (14)$$

Since $\psi_{1_{(0)}}^e = 1 = \psi_{2_{(h_e)}}^e$. Above $F_i^e = \left(\int_0^{h_e} f_{(\bar{x})} \psi_i^e \, d\bar{x} \right)$ is a vector of distributed forces, and recall that $P_1^e = -E\,A \left.\frac{\partial u}{\partial \bar{x}}\right|_{\bar{x}=0}$; $P_2^e = E\,A \left.\frac{\partial u}{\partial \bar{x}}\right|_{\bar{x}=h_e}$ are the nodal reaction axial forces.

Substitution of Eqs. (11, 14) into the **Hamiltonian Principle** in Eq. (8) leads to the system of equations for **ONE element**:

$$\sum_{j=1}^{n} M_{ij}^e \ddot{u}_j^e + \sum_{j=1}^{n} K_{ij}^e u_j^e = F_i^e + P_i^e \qquad (15a)$$

or in matrix form

$$\mathbf{M^e \ddot{u}^e + K^e u = F^e + P^e} \qquad (15b)$$

From Eqs. (13) the shape functions $\left\{\psi_i^e\right\}_{i=1,2}$ must satisfy the **essential boundary conditions**

$$\psi^e_{1_{(\bar{x}=0)}} = 1,\ \psi^e_{1_{(\bar{x}=h_e)}} = 0;$$

$$\psi^e_{2_{(\bar{x}=0)}} = 0,\ \psi^e_{1_{(\bar{x}=h_e)}} = 1$$

and $$\left\{\psi_i^e\right\}_{i=1,2} \in \mathrm{C}^1\left[0,h_e\right]$$

Select, $$\psi_1^e = \left(1-\frac{\bar{x}}{h_e}\right);\psi_2^e = \left(\frac{\bar{x}}{h_e}\right) \qquad (16)$$

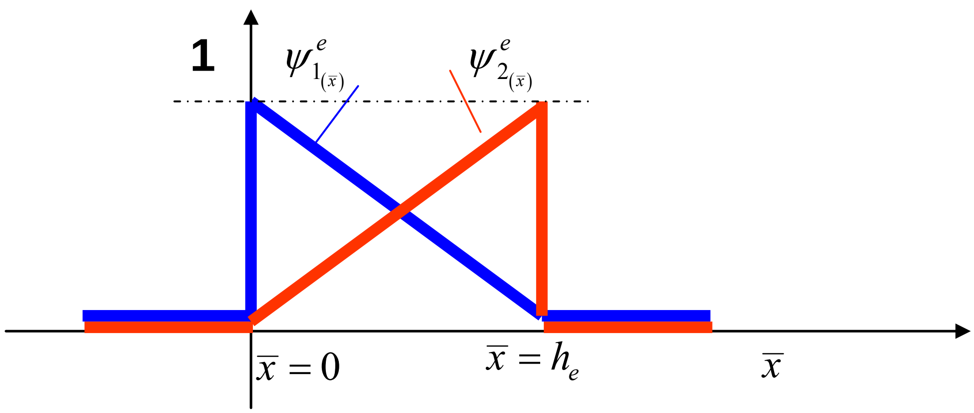


Fig 5. Shape functions for a finite element in axial bar

Note that

a) ψ_1 and ψ_2 are linear combinations of the linearly independent complete set $\{1,\bar{x}\}$. In addition, $\left(\psi_1^e+\psi_2^e\right)=1$, the shape functions are a partition of unity (This means the shape functions $\left\{\psi_i^e\right\}_{i=1,2}$ are able to model rigid body displacements).

b) the set $\left\{\psi_i^e\right\}_{i=1,2}$ is different from zero only on Ω^e and elsewhere is zero. This quality is called a **local support** and it is extremely important to make the global **M and K** matrices **banded**, i.e., with a small number of non-zero values.

The derivation of the shape (or interpolation) functions $\left\{\psi_i^e\right\}_{i=1,2}$ does not depend on the problem. The functions do depend on the type of element (geometry, number of nodes or joints, and the number of primary unknowns).

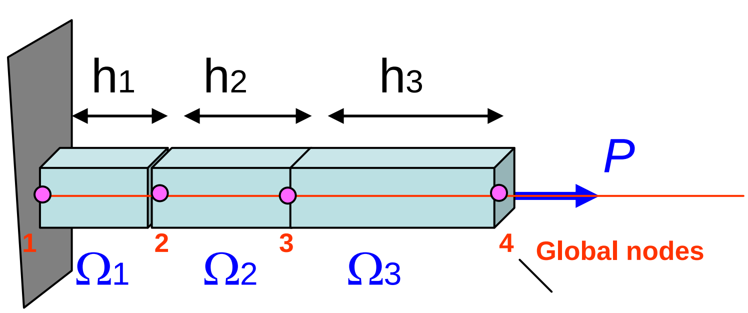
Substitution of the shape functions, Eq. (16), into the **element** mass and stiffness matrices gives:

$$\mathbf{M^e} = \frac{\rho_e A_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; \mathbf{K^e} = \frac{A_e E_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \mathbf{F^e} = \frac{f_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad (17)$$

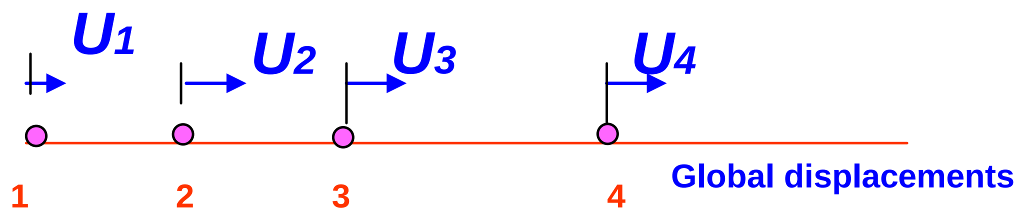
for a bar with uniform cross sectional area (A_e) and material properties $(\rho, E)_e$ within the element. Note that $\mathbf{K^e}$ is a singular matrix.

The next step constructs (in the computer) the matrices $(\mathbf{M^e}, \mathbf{K^e})_{e=1,2,..Ne}$ for each element in the domain of interest. Next, one performs the interconnection or assembly of the element EOMs.

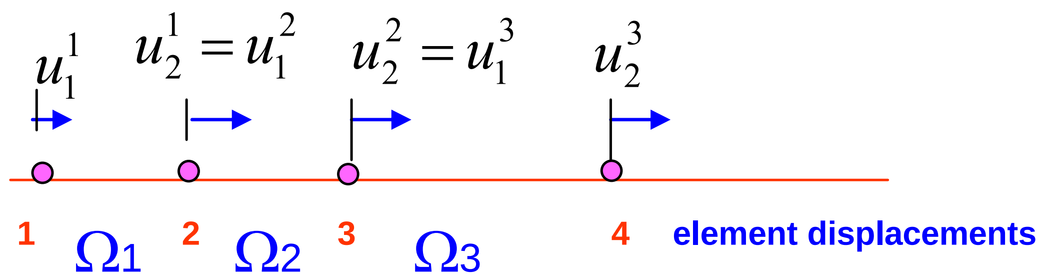
For the sake of discussion, suppose the domain $\Omega = (0, L)$ is divided into 3 elements of possibly unequal lengths, as shown below



Define the **global displacements** **U**=$\{U_1, U_2, U_3, U_4\}$



while the (local) nodal displacements in each element (e) are:



The elements are connected at *global nodes* (2) and (3); since the displacement **U** needs to be continuous (i.e., without cracks, fractures, etc.). Then, the **interelement** continuity conditions are

$$\begin{aligned} U_1 &= u_1^1, \\ U_2 &= u_2^1 = u_1^2, \\ U_3 &= u_2^2 = u_1^3, \\ U_4 &= u_2^3 \end{aligned} \tag{18}$$

A BOOLEAN or CONNECTIVITY ARRAY (B) states the correspondence between **local nodes** in an element and the **global nodes.**

$B_{ij} \equiv$ the **global node** number corresponding to the j-th node of element i
$i = 1,2 \ldots .N_e$: number elements on mesh.
$j = 1,2 \ldots .N_n$: number of nodes per element.

Presently, $$\mathbf{B}=\begin{bmatrix}1\,2\\2\,3\\3\,4\end{bmatrix}$$

Repetition of a number in **B** indicates that the coefficients of $\mathbf{K^e}$ and $\mathbf{M^e}$ associated with the node number will add.

In a computer implementation of the FE scheme, the connectivity array is used extensively for the **automatic assembly** of the global system of equations.

For the three elements in the example, the element Eqns. (17) are written in global coordinates or nodes as:

e=1:
$$\frac{\rho A h_1}{6}\begin{bmatrix}2&1&0&0\\1&2&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{Bmatrix}\ddot{U}_1\\\ddot{U}_2\\\ddot{U}_3\\\ddot{U}_4\end{Bmatrix}+\frac{AE}{h_1}\begin{bmatrix}1&-1&0&0\\-1&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{Bmatrix}U_1\\U_2\\U_3\\U_4\end{Bmatrix}=\begin{Bmatrix}F_1^1\\F_2^1\\0\\0\end{Bmatrix}+\begin{Bmatrix}P_1^1\\P_2^1\\0\\0\end{Bmatrix}$$

e=2:
$$\frac{\rho A h_2}{6}\begin{bmatrix}0&0&0&0\\0&2&1&0\\0&1&2&0\\0&0&0&0\end{bmatrix}\begin{Bmatrix}\ddot{U}_1\\\ddot{U}_2\\\ddot{U}_3\\\ddot{U}_4\end{Bmatrix}+\frac{AE}{h_2}\begin{bmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{bmatrix}\begin{Bmatrix}U_1\\U_2\\U_3\\U_4\end{Bmatrix}=\begin{Bmatrix}0\\F_1^2\\F_2^2\\0\end{Bmatrix}+\begin{Bmatrix}0\\P_1^2\\P_2^2\\0\end{Bmatrix}$$

e=3:
$$\frac{\rho A h_3}{6}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&2&1\\0&0&1&2\end{bmatrix}\begin{Bmatrix}\ddot{U}_1\\\ddot{U}_2\\\ddot{U}_3\\\ddot{U}_4\end{Bmatrix}+\frac{AE}{h_2}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&-1\\0&0&-1&1\end{bmatrix}\begin{Bmatrix}U_1\\U_2\\U_3\\U_4\end{Bmatrix}=\begin{Bmatrix}0\\0\\F_1^3\\F_2^3\end{Bmatrix}+\begin{Bmatrix}0\\0\\P_1^3\\P_2^3\end{Bmatrix}$$

The global EOMS for the whole system are obtained by superposition (addition) of the equations above:

$$\mathbf{M\ddot{U}} + \mathbf{K\,U} = \mathbf{F} + \mathbf{P} = \begin{Bmatrix} F_1^1 \\ F_2^1 + F_1^2 \\ F_2^2 + F_1^3 \\ F_2^3 \end{Bmatrix} + \begin{Bmatrix} P_1^1 \\ P_2^1 + P_1^2 \\ P_2^2 + P_1^3 \\ P_2^3 \end{Bmatrix} \tag{18}$$

where

$$\mathbf{M} = \bigcup_{e=1}^{N_e}\left(\mathbf{M^e}\right); \mathbf{K} = \bigcup_{e=1}^{N_e}\left(\mathbf{K^e}\right) \tag{19}$$

are the global matrices of mass and stiffness coefficients, and

$$\mathbf{F} = \bigcup_{e=1}^{N_e}\left(\mathbf{f^e}\right); \mathbf{P} = \bigcup_{e=1}^{N_e}\left(\mathbf{P^e}\right) \tag{20}$$

are the vectors of distributed forces and nodal forces, respectively.

Note above **K** is singular since the boundary condition (fixed end at x=0, i.e. U_1=0) is yet to be applied.

Imposition of boundary conditions

In general, due to continuity (action=reaction), at the internal nodes

$$P_2^1 + P_1^2 = 0,\; P_2^2 + P_1^3 = 0$$

Or $$P_2^1 + P_1^2 \leftarrow Q_2,\; P_2^2 + P_1^3 \leftarrow Q_3,\; P_2^3 \leftarrow Q_4 \tag{21}$$

with Q_2, Q_3 and Q_4 as specified nodal (axial) forces acting on the bar.

Incidentally, U_1=0 is specified, while the wall reaction force $P_1^1 = ?$ is an unknown.

In general, let $\mathbf{U} = \begin{Bmatrix} \mathbf{U_a} \\ \mathbf{U_d} \end{Bmatrix}$, where $\mathbf{U_a}$ is a vector containing the active DOF (degrees of freedom) and $\mathbf{U_d}$ are the specified **(time invariant)** DOF. Then, the global equations of motion are partitioned as

$$\begin{bmatrix} \mathbf{M_{aa}} & \mathbf{M_{ad}} \\ \mathbf{M_{da}} & \mathbf{M_{dd}} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{U}}_\mathbf{a} \\ \ddot{\mathbf{U}}_\mathbf{d} \end{Bmatrix} + \begin{bmatrix} \mathbf{K_{aa}} & \mathbf{K_{ad}} \\ \mathbf{K_{da}} & \mathbf{K_{dd}} \end{bmatrix} \begin{Bmatrix} \mathbf{U_a} \\ \mathbf{U_d} \end{Bmatrix} = \begin{Bmatrix} \mathbf{S_a} \\ \mathbf{S_d} \end{Bmatrix} \quad (22)$$

where $\mathbf{M_{ad}} = \mathbf{M_{da}}, \mathbf{K_{da}} = \mathbf{K_{ad}}$. $\mathbf{S_a} = \mathbf{F} + \mathbf{Q}$ is a vector of (known) applied forces (distributed and nodal) and $\mathbf{S_d}$ is a vector of <u>unknown</u> reaction forces (to be calculated)

Since $\ddot{\mathbf{U}}_\mathbf{d} = \mathbf{0}$, expand Eqs. (22) as

$$\mathbf{M_{aa}} \ddot{\mathbf{U}}_\mathbf{a} + \mathbf{K_{aa}} \mathbf{U_a} + \mathbf{K_{ad}} \mathbf{U_d} = \mathbf{S_a}$$
$$\mathbf{M_{da}} \ddot{\mathbf{U}}_\mathbf{a} + \mathbf{K_{da}} \mathbf{U_a} + \mathbf{K_{dd}} \mathbf{U_d} = \mathbf{S_d}$$

Solve for $\mathbf{U_a}$ from the first equation above,

$$\mathbf{M_{aa}} \ddot{\mathbf{U}}_\mathbf{a} + \mathbf{K_{aa}} \mathbf{U_a} = \mathbf{S_a} - \mathbf{K_{ad}} \mathbf{U_d} \quad (23)$$

and next, find the internal forces $\mathbf{S_d}$ from the 2nd Equation

$$\mathbf{S_d} \leftarrow \mathbf{M_{da}} \ddot{\mathbf{U}}_\mathbf{a} + \mathbf{K_{da}} \mathbf{U_a} + \mathbf{K_{dd}} \mathbf{U_d} \quad (24)$$

For the elastic bar, the system of eqns. (23) is **<u>tridiagonal</u>** and thus, its solution can be obtained quickly without a matrix inversion.

Note that the essential BCs is a specified $U_1 = 0$ while the reaction force $P_1^{\,1}$ is unknown. Hence, the final EOMS are:

$$\begin{bmatrix} M_{2,2} & M_{2,3} & M_{2,4} \\ M_{2,3} & M_{3,3} & M_{3,4} \\ M_{4,2} & M_{3,4} & M_{4,4} \end{bmatrix} \begin{Bmatrix} \ddot{U}_2 \\ \ddot{U}_3 \\ \ddot{U}_4 \end{Bmatrix} + \begin{bmatrix} K_{2,2} & K_{2,3} & K_{2,4} \\ K_{2,3} & K_{3,3} & K_{3,4} \\ K_{4,2} & K_{3,4} & K_{4,4} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P_{(t)} \end{Bmatrix} \qquad (25)$$

and once the equation above is solved,

$$M_{1,2}\ddot{U}_2 + K_{1,2}U_2 + M_{1,3}\ddot{U}_3 + K_{1,3}U_3 + M_{1,4}\ddot{U}_4 + K_{1,4}U_4 \rightarrow P_1^1 \quad (26)$$

In the example configuration, the satisfaction of the essential constant $U_1 = 0$ removes the singularity of the stiffness matrix (i.e., removes the rigid body mode).

For the example case, considering elements of equal length, $h_e = L/3$, the equations of motion are:

$$\frac{\rho AL}{18} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{U}_2 \\ \ddot{U}_3 \\ \ddot{U}_4 \end{Bmatrix} + \frac{3AE}{L} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P_{(t)} \end{Bmatrix} \qquad (27)$$

Use **MODAL ANALYSIS** to solve

$$\overline{\mathbf{M}}\,\ddot{\mathbf{U}} + \overline{\mathbf{K}}\,\mathbf{U} = \overline{\mathbf{F}} \tag{28}$$

+Initial Conditions

To this end, **first s**olve the homogeneous EOMs

$$\overline{\mathbf{M}}\,\ddot{\mathbf{U}} + \overline{\mathbf{K}}\,\mathbf{U} = \mathbf{0} \rightarrow \left[\,\overline{\mathbf{K}} - \omega_i^2\,\overline{\mathbf{M}}\,\right]{}^i\phi = \mathbf{0} \tag{29}$$

and make the modal matrix, $\Phi = \left[\,{}^1\phi\ {}^2\phi\ldots\ldots{}^N\phi\,\right]$ (30)

where N is the active degrees of freedom in the system.

Using the modal transformation $\mathbf{U}_{(t)} = \Phi\,\eta_{(t)}$, the physical EOMS become in modal coordinates:

$$\mathbf{M_m}\,\ddot{\eta} + \mathbf{K_m}\,\eta = \mathbf{Q_m} \tag{31}$$

where $\mathbf{M_m} = \Phi^T\overline{\mathbf{M}}\Phi;\quad \mathbf{K_m} = \Phi^T\overline{\mathbf{K}}\Phi$, and $\mathbf{Q_m} = \Phi^T\overline{\mathbf{F}}$ (32)

are the modal mass and stiffness matrices (diagonal) and the modal force vector, respectively.

By now, you do know how to solve the uncoupled set of N ODES and then return to physical coordinates.

Examples – How good is a FE model ?

Recall the EOM for one-element

$$\mathbf{M^e}\,\ddot{\mathbf{U}}^{\mathbf{e}} + \mathbf{K^e}\mathbf{U^e} = \mathbf{F^e} \tag{a1}$$

where

$$\mathbf{U^e} = \{U_1 \quad U_2\}^T ; \mathbf{F^e} = \{F_1 \quad F_2\}^T \tag{a2}$$

and

$$\mathbf{M^e} = \frac{\rho A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \mathbf{K^e} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{a3}$$

with L as the bar length with cross-section A and material properties (ρ,E).

Bar with one end fixed: Let's find its fundamental nat. frequency

Set

$$\mathbf{U^e} = \{U_1 = 0 \quad U_2\}^T ; \mathbf{F^e} = \{F_1 = ? \quad F_2 = 0\}^T$$

EOM (a1) reduces to

$$\frac{\rho A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} 0 \\ \ddot{U}_2 \end{Bmatrix} + \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

Or $\rightarrow \left(\frac{\rho A L}{3}\right)\ddot{U}_2 + \left(\frac{AE}{L}\right)U_2 = 0$

Then $\omega_{1\_FEM} = \frac{\sqrt{3}}{L}\left(\frac{E}{\rho}\right)^{1/2}$ and $\omega_1 = \frac{\pi}{2L}\left(\frac{E}{\rho}\right)^{1/2}$ (exact) HD14

The ratio of natural frequencies FEM/exact = **1.102**

Bar with two free ends: Find its fundamental elastic mode natural frequency

Set $\mathbf{U^e} = \{U_1 \quad U_2\}^T; \mathbf{F^e} = \{F_1 = 0 \quad F_2 = 0\}^T$

EOM (a1) reduces to

$$\frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{Bmatrix} + \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Process 2DOF equation above to obtain

$$\omega_{1\_FEM} = 0, \quad \omega_{2\_FEM} = \frac{\sqrt{12}}{L}\left(\frac{E}{\rho}\right)^{1/2}$$

$$\omega_1 = 0, \qquad \omega_2 = \frac{\pi}{L}\left(\frac{E}{\rho}\right)^{1/2} \quad \text{(exact)}$$ Handout 14

The ratio of elastic mode nat. frequencies FEM/exact= **1.102**

ONE FE model delivers close predictions for the elastic natural frequencies.

A question to ponder:
Why does the FE model produces LARGER natural frequencies than the exact (analytical) ones?

A FE model for bending of elastic beams

Figure 6 depicts an elastic cantilever beam with lateral forces (F) and moments (M) acting on it that produce the beam lateral deflection or bending $v_{(x,t)}$. F_o and M_o are a lateral force and moment applied at the end of the beam (x=L), and $f_{(x)}$ is a distributed force/unit length. The beam has material density (ρ) and elastic modulus (E) and geometric properties of area (A) and area moment of inertia (I).

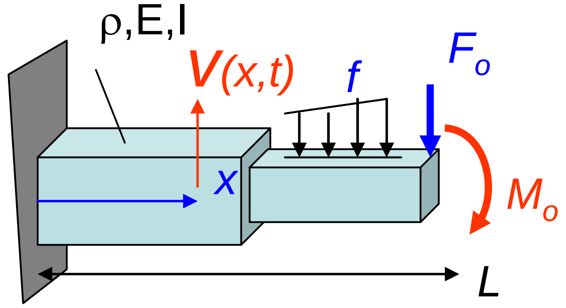


Fig 6. Elastic beam under lateral (bending) displacements

For a finite element (Ω^e) or piece of the beam with width h, note the sign convention:

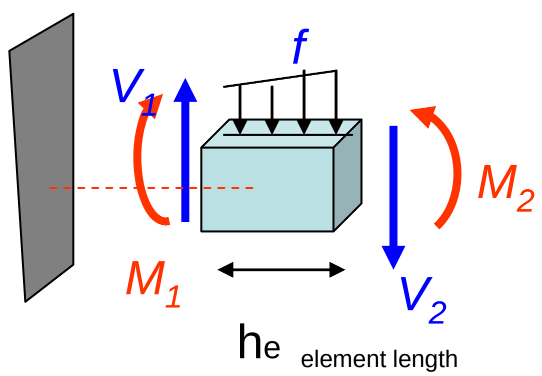


Fig 7. Notation for shear forces (V) and moments (M)

Let V and M denote the shear force and bending moment. For an Euler-type beam

$$M = EI\frac{\partial^2 v}{\partial x^2};\; V = \frac{\partial M}{\partial x} \tag{33}$$

In the FEM, discretize the beam (Ω) into a collection of finite elements $\Omega^e \left(\Omega = \cup\Omega^e\right)$, as shown below in Fig. 8.

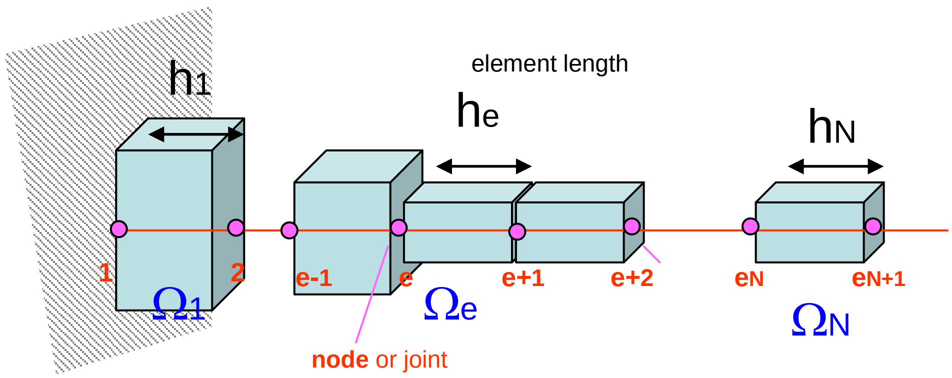


Fig 8. Discretization of beam into *N* finite elements

Let v and $\theta = \partial v / \partial x$ be the lateral displacement and beam rotation (*primary* and *secondary* variables). Fig. 9 shows a free body diagram in the element Ω^e, with $x_{e+1} = x_e + h_e$,

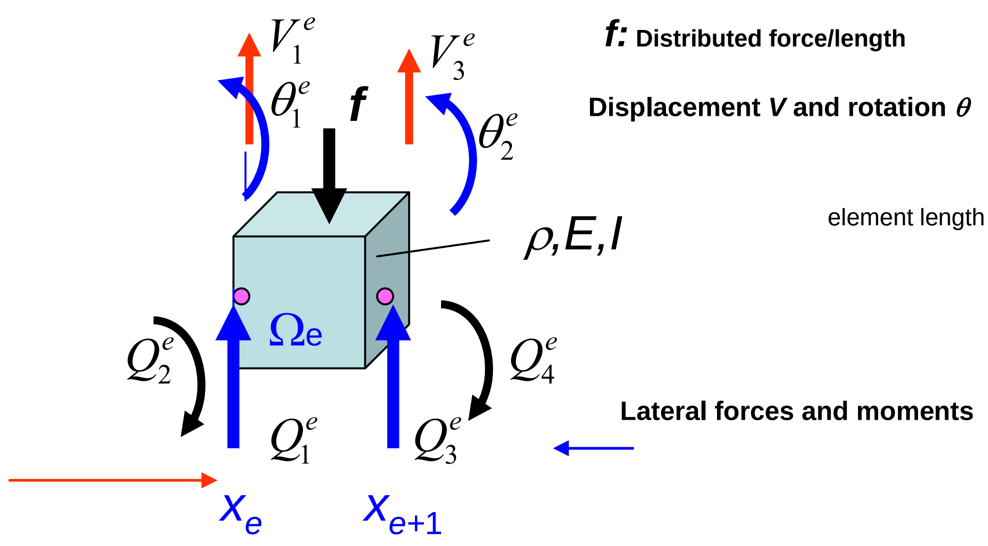


Fig 9. Free body diagram for a finite element in a elastic beam

The **shear forces** at the ends of the element

$$Q_1^e = V_{(x_e)} = \frac{\partial}{\partial x}\left(E\, I \frac{\partial^2 v}{\partial x^2}\right]_{x_e} ;$$

$$Q_3^e = V_{(x_{e+1})} = -\frac{\partial}{\partial x}\left(E\, I \frac{\partial^2 v}{\partial x^2}\right]_{x_{e+1}} \tag{34}$$

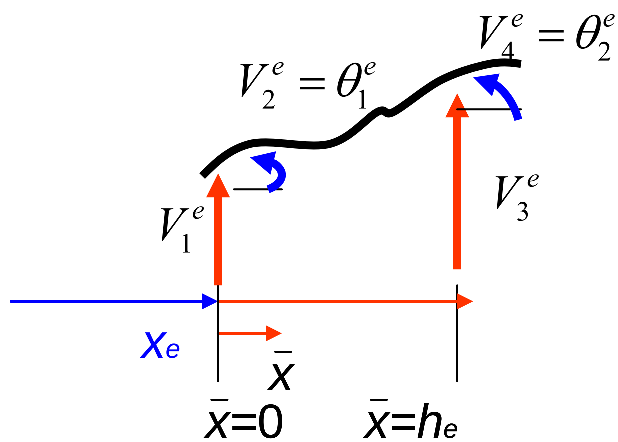
And the **bending moments** at the ends are

$$Q_2^e = M_{(x_e)} = \left(E\, I \frac{\partial^2 v}{\partial x^2}\right]_{x_e}$$

$$Q_4^6 = M_{(x_{e+1})} = -\left(E\, I \frac{\partial^2 v}{\partial x^2}\right]_{x_{e+1}} \tag{35}$$

Let

$$V_1^e = v_{(x_e,t)};\; V_3^e = v_{(x_{e+1},t)}$$
$$V_2^e = \theta_{(x_e,t)} = \theta_1^e;\; V_4^e = \theta_{(x_{e+1},t)} = \theta_2^e \qquad (36)$$



$\bar{x} = x - x_e$

Assume that the beam lateral displacement and rotation (angle) in Ω^e are given by the approximation

$$v^e_{(x,t)} = \sum_{i=1}^{4} \psi^e_{i_{(\bar{x})}} V^e_{i_{(t)}}$$
$$\theta^e_{(x,t)} = \sum_{i=1}^{4} \frac{d\psi^e_{i_{(\bar{x})}}}{d\bar{x}} V^e_{i_{(t)}} \qquad (37)$$

From a prior discussion on the **assumed modes method**, the elements of the mass and stiffness matrices for Ω^e are

$$M^e_{ij} = M^e_{ji} = \int_0^{h_e} \rho^e A^e \psi^e_i \psi^e_j \, d\bar{x},$$
$$K^e_{ij} = K^e_{ji} = \int_0^{h_e} E^e I^e \frac{d^2\psi^e_i}{d\bar{x}^2} \frac{d^2\psi^e_j}{d\bar{x}^2} d\bar{x} \qquad (38)$$

Note that $\left\{\psi_i^e\right\}_{i=1,..4} \in C^2\left(\Omega^e\right)$. The shape functions satisfy the essential boundary conditions,

$$
\begin{aligned}
v_{(x_e,t)} &= V_1^e \rightarrow \psi^e_{1_{(0)}} = 1;\ \psi^e_{2_{(0)}} = \psi^e_{3_{(0)}} = \psi^e_{4_{(0)}} = 0 \\
v_{(x_{e+1},t)} &= V_3^e \rightarrow \psi^e_{3_{(h_e)}} = 1;\ \psi^e_{1_{(h_e)}} = \psi^e_{3_{(h_e)}} = \psi^e_{4_{(h_e)}} = 0
\end{aligned} \tag{39a}
$$

$$
\begin{aligned}
\theta_{(x_e,t)} &= V_2^e \rightarrow d\psi^e_{2(0)}/d\bar{x} = 1;\ d\psi^e_{1(0)}/d\bar{x} = d\psi^e_{3(0)}/d\bar{x} = d\psi^e_{4(0)}/d\bar{x} = 0 \\
v_{(x_{e+1},t)} &= V_3^e \rightarrow d\psi^e_{4(h_e)}/d\bar{x} = 1;\ d\psi^e_{1(h_e)}/d\bar{x} = d\psi^e_{2(h_e)}/d\bar{x} = d\psi^e_{3(h_e)}/d\bar{x} = 0
\end{aligned} \tag{39b}
$$

The lowest order polynomial that satisfies the conditions above **is third order** (and is twice differentiable) , i.e.,

$$
\begin{aligned}
v_{(\bar{x},t)} &= c_0 + c_1\left(\frac{\bar{x}}{h_e}\right) + c_2\left(\frac{\bar{x}}{h_e}\right)^2 + c_3\left(\frac{\bar{x}}{h_e}\right)^3 \\
\theta_{(\bar{x},t)} &= + c_1\left(\frac{1}{h_e}\right) + 2c_2\left(\frac{\bar{x}}{h_e^2}\right) + 3c_3\left(\frac{\bar{x}^2}{h_e^3}\right)
\end{aligned} \tag{40}
$$

Leading to the following conditions,

$$
\begin{aligned}
&\bar{x} = 0:\ V_1^e = c_0,\ V_2^e = \theta_1 = \left(\frac{c_1}{h_e}\right) \\
&\bar{x} = h_e:\ V_3^e = c_0 + c_1 + c_2 + c_3, \\
&\qquad V_4^e = \theta_2 = \left(\frac{c_1}{h_e}\right) + \left(\frac{2c_2}{h_e}\right) + \left(\frac{3c_3}{h_e}\right)
\end{aligned} \tag{41}
$$

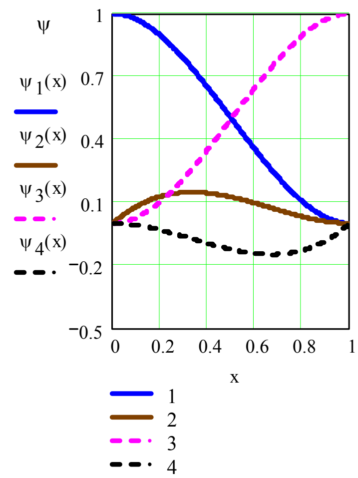
Solving the set of four Eqs. (41) gives

$$\psi_1^e = 1 - 3\left(\frac{\bar{x}}{h_e}\right)^2 + 2\left(\frac{\bar{x}}{h_e}\right)^3$$

$$\psi_2^e = \bar{x}\left(1 - \frac{\bar{x}}{h_e}\right)^2$$

$$\psi_3^e = 3\left(\frac{\bar{x}}{h_e}\right)^2 - 2\left(\frac{\bar{x}}{h_e}\right)^3$$

$$\psi_4^e = \frac{\bar{x}^2}{h_e}\left(\frac{\bar{x}}{h_e} - 1\right)$$



(42)

Figure 10 depicts the shape functions. Note that $\left(\psi_1^e + \psi_2^e + \psi_3^e + \psi_4^e\right) = 1$

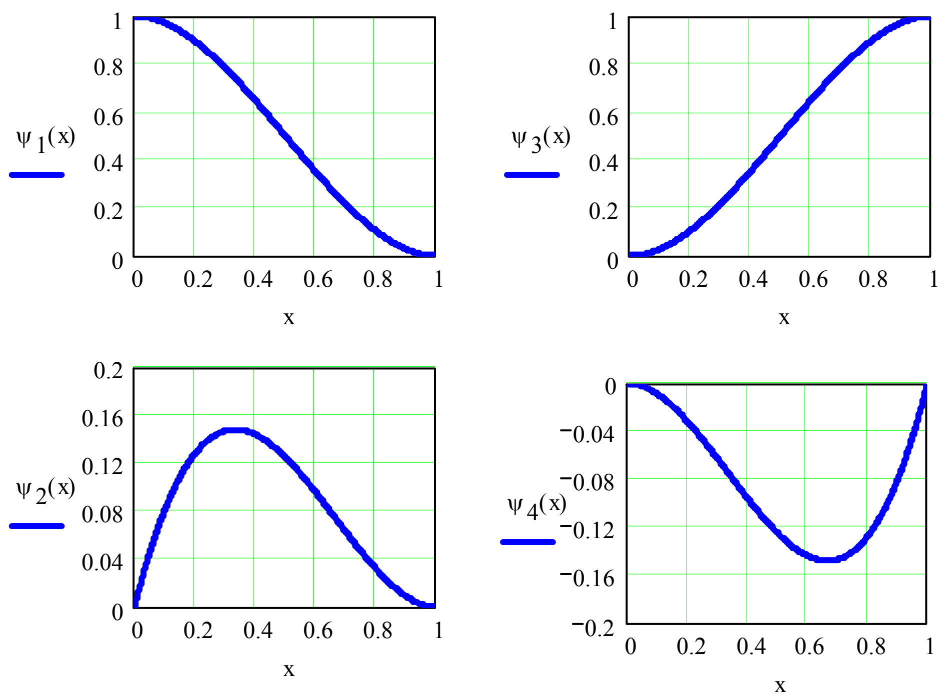


Fig 10. Shape functions ψ for structural element under bending.

Substitution of Eq. (42) into Eq. (38), renders the following element mass and stiffness matrices.

$$\mathbf{M^e} = \frac{\rho A h_e}{420}\begin{bmatrix} 156 & 22h_e & 54 & -13h_e \\ 22h_e & 4h_e^2 & 13h_e & 2h_e^2 \\ 54 & 13h_e & 156 & -22h_e \\ -13h_e & 2h_e^2 & -22h_e & 4h_e^2 \end{bmatrix} = \left(\mathbf{M^e}\right)^T \tag{43}$$

$$\mathbf{K^e} = \frac{E^e I^e}{h_e^3}\begin{bmatrix} 12 & 6h_e & -12 & 6h_e \\ 6h_e & 4h_e^2 & -6h_e & 2h_e^2 \\ -12 & -6h_e & 12 & -6h_e \\ 6h_e & 2h_e^2 & -6h_e & 4h_e^2 \end{bmatrix} = \left(\mathbf{K^e}\right)^T \tag{44}$$

and the vector of generalized forces $\{F_i^e\}$ is:

$$F_i^e = \int_0^{h_e} f_{(\bar{x},t)} \psi_{i_{(\bar{x},t)}} d\bar{x} \tag{45}$$

Assuming a constant distributed force $f_{(x,t)}$ over the element, obtain:

$$\mathbf{F^e} = \left\{ \tfrac{1}{2} f_e h_e \quad \tfrac{1}{12} f_e h_e^2 \quad \tfrac{1}{2} f_e h_e \quad \tfrac{1}{12} f_e h_e^2 \right\}^T \tag{45b}$$

and the constraint nodal forces are obtained from the *virtual* work performed:

$$\delta W = Q_1^e \delta V_1^e + Q_3^e \delta V_3^e - Q_2^e \delta V_2^e - Q_4^e \delta V_4^e \tag{46}$$

Hence,

$$\mathbf{Q^e} = \left\{ Q_1^e \quad -Q_2^e \quad Q_3^e \quad -Q_4^e \right\}^T \tag{46b}$$

Then, the system of equations for the element Ω^e are:

$$\sum_{j=1}^{4} M_{ij}^{e}\ddot{V}_{j}^{e} + \sum_{j=1}^{4} K_{ij}^{e}V_{j}^{e} = F_{i}^{e} + Q_{i}^{e} \qquad (47)$$

or

$$\mathbf{M^e}\mathbf{\ddot{V}^e} + \mathbf{K^e}\mathbf{V}^e = \mathbf{F^e} + \mathbf{Q^e} \qquad (47)$$

where

$$\mathbf{V^e} = \left\{ V_1^e \quad V_2^e \quad V_3^e \quad V_4^e \right\}^T$$

Assembly of the element matrices to produce the **global system** of equations is easily done as exemplified before. Be careful to keep continuity of the displacements at the nodes (joints) and also the continuity of constraint forces at the nodes.

The global system of equations becomes

$$\mathbf{M\,\ddot{V} + K\,V = F + Q} \qquad (48)$$

where the global vector of displacements is

$$\mathbf{V} = \left\{ V_1, V_2 = \theta_1, V_3, V_4 = \theta_2, \ldots\ldots\ldots\ldots V_N, V_N = \theta_N \right\}^T \qquad (49)$$

with N as the global number of nodes);

$$\mathbf{M} = \bigcup_{e=1}^{N_e}\left(\mathbf{M^e}\right); \mathbf{K} = \bigcup_{e=1}^{N_e}\left(\mathbf{K^e}\right) \qquad (50)$$

are the global matrices of mass and stiffness coefficients, and

$$\mathbf{F} = \bigcup_{e=1}^{N_e}\left(\mathbf{f^e}\right); \mathbf{Q} = \bigcup_{e=1}^{N_e}\left(\mathbf{Q^e}\right) \qquad (51)$$

are the vectors of distributed forces and nodal forces, respectively.

Note above **K** is singular since the boundary conditions are yet to be applied.

The global FEM vector of forces **Q** will in general have zero components at the internal nodes. The elements of this vector are of the form, see Fig. 11:

$$Q_k = Q_3^e + Q_1^{e+1}; \quad Q_{k+1} = Q_4^e + Q_2^{e+1} \tag{52}$$

Here *k* denotes the global node number.

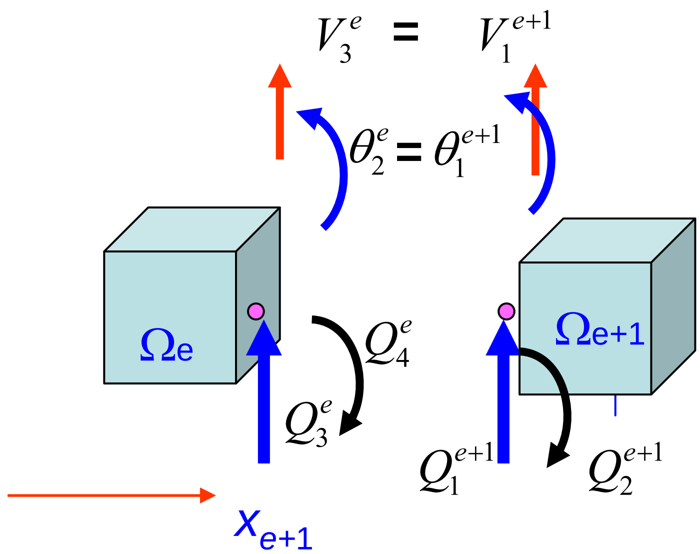


Fig 11. Balance of forces at a node

The eqns. above constitute a statement of equilibrium of forces at the nodal interface (boundary) of an element. Recall from Eqns. (34, 35) that

$$Q_3^e + Q_1^{e+1} = -\frac{\partial}{\partial x}\left(E\,I \frac{\partial^2 v}{\partial x^2}\right]_{x_{e+1}} + \frac{\partial}{\partial x}\left(E\,I \frac{\partial^2 v}{\partial x^2}\right]_{x_{e+1}} = 0$$

$$Q_4^e + Q_2^{e+1} = -\left(E\,I \frac{\partial^2 v}{\partial x^2}\right]_{x_{e+1}} + \left(E\,I \frac{\partial^2 v}{\partial x^2}\right]_{x_{e+1}} = 0$$

If no external forces or moments are applied at the node, then

$$Q_k = Q_3^e + Q_1^{e+1} = 0,$$
$$Q_{k+1} = Q_4^e + Q_2^{e+1} = 0 \qquad (53)$$

However, if an external nodal shear force or moment is applied at the joint, then the components of the force vector **Q** will not be zero.

Consider, for example, the case of a **support spring with stiffness k_s connecting the beam to ground**, i.e.,

$$Q_3^e + Q_1^{e+1} + F_{spring} = 0;$$

$$Q_k = Q_3^e + Q_1^{e+1} = -F_{spring} = -k_s V_1^{e+1} \qquad (54)$$

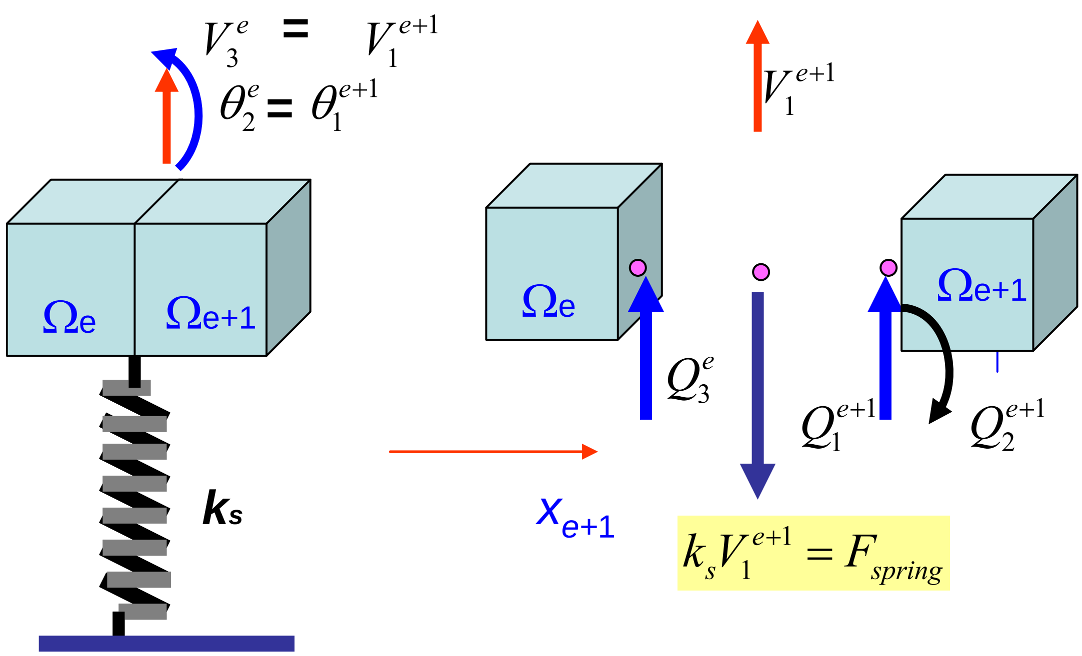


Fig 12. Balance of forces at a node with a spring connection

Note that this spring restoring force depends on the element lateral deflection, and therefore, it is unknown. Hence, one needs to modify the global stiffness matrix and add the contribution of the support stiffness k_{s}..

Coordinate transformation of element matrices:

A **plane elastic frame** element combines the elastic properties of a bar and a beam; and hence, it has three displacement coordinates at each end (two orthogonal displacements and one rotation). As shown in Figure 13, this element has a local coordinate system (x,y) where the x-coordinate aligns with the major axis (length) of the bar-beam.

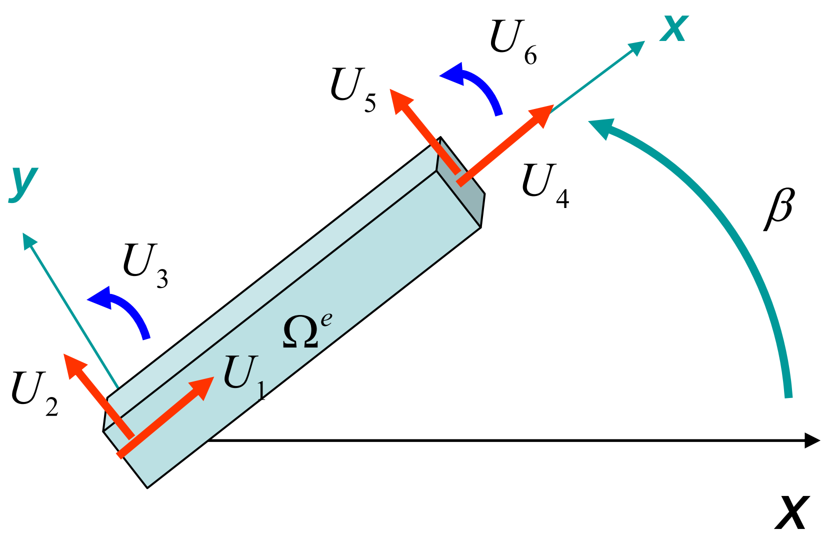


Fig 13. Arbitrary frame element and displacements

The x-axis is tilted angle β relative to a global (inertial) coordinate system (X, Y) to which all elements in the structure will be related. In the (X,Y) coordinate system, the displacements are defined as shown in Fig. 14.

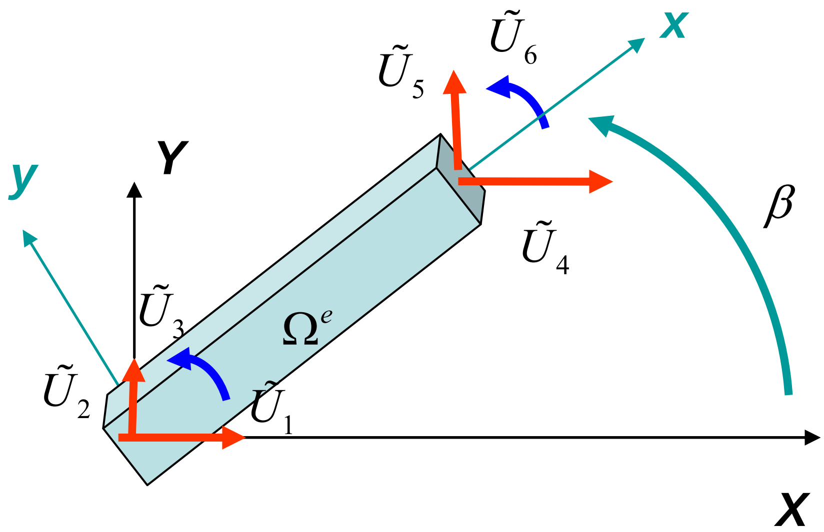


Fig 14. Arbitrary frame element and displacements in (X,Y) axes

The transformation between the local displacements $\left\{\tilde{V}_i^e\right\}_{i=1,2...6}$ in the (X,Y) to the displacements $\left\{V_i^e\right\}_{i=1,2...6}$ in the local coordinate system (x,y) is given by the **(coordinate transformation)** equation

$$\mathbf{V^e} = \mathbf{T_e}\tilde{\mathbf{V}}^\mathbf{e} \tag{55}$$

where

$$\mathbf{T_e} = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 & 0 & 0 \\ -\sin\beta & \cos\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\beta & \sin\beta & 0 \\ 0 & 0 & 0 & -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{56}$$

The element EOMs in the (x,y) coordinate system are

$$\mathbf{M^e}\ddot{\mathbf{V}}^\mathbf{e} + \mathbf{K^e}\mathbf{V}^e = \mathbf{F^e} + \mathbf{Q^e} \tag{57}$$

Substitution of (Eq. 55) into (57) and pre-multiplying by $\mathbf{T_e}^\mathbf{T}$ gives the following:

$$\mathbf{T_e^T M^e T_e \ddot{\tilde{V}}^e + T_e^T K^e T_e \tilde{V}^e = T_e^T F^e + T_e^T Q^e} \qquad (58)$$

Let: $\mathbf{\tilde{M}^e = T_e^T M^e T_e}$, $\mathbf{\tilde{K}^e = T_e^T K^e T_e}$, $\mathbf{\tilde{F}^e = T_e^T F^e}$, $\mathbf{\tilde{Q}^e = T_e^T Q^e}$ (59)

and write the EOM as: $\mathbf{\tilde{M}^e \ddot{\tilde{V}}^e + \tilde{K}^e \tilde{V}^e = \tilde{F}^e + \tilde{Q}^e}$ (60)

Note that the resulting mass and stiffness matrices are still symmetric.

The assembly of the element matrices proceeds in the usual way to obtain the global system of equations.

Constraints and reduction of degrees of freedom:

Thus far, the analysis takes all generalized displacements as independent of each other. The assumption lead to the global system of equations:

$$\mathbf{M \ddot{V} + K V = F + Q} \qquad (61)$$

Frequently, there arises a need to specify relationships amongst system displacement coordinates. This is equivalent to specifying $\widehat{N}$ displacements which are linearly independent and the rest $\widehat{N}+1$, $\widehat{N}+2$,….N depend on the $\widehat{N}$ displacements.

The discussion presently relates to a constraint equation of the form:

$$f_i\left(V_{\widehat{N}+1}, V_{\widehat{N}+2}, \ldots V_N\right) - g_i\left(V_1, V_2, \ldots V_{\widehat{N}}\right) = 0 \quad {}_{i=\widehat{N}+1,\ldots,N} \qquad (62)$$

The equation above can be written in matrix form as:

$$\mathbf{R\,V} = \left[\mathbf{R_{da}} \quad \mathbf{R_{dd}}\right]\begin{Bmatrix}\mathbf{V_a}\\ \mathbf{V_d}\end{Bmatrix} = \mathbf{0} \qquad (63)$$

where $\mathbf{V_d}$ is the vector of N_d dependent coordinates or displacements, and $\mathbf{V_a}$ is the vector of $\hat{N} = N_a =$ N_a <u>independent</u> or ACTIVE coordinates or displacements. Note $\hat{N} + N_d = N_a + N_d = N$.

From eq. (63) $\mathbf{R_{da} V_a + R_{dd} V_d = 0}$ find

$$\rightarrow \quad \mathbf{V_d = T_{da} V_a = \left(-R_{dd}^{-1} R_{da}\right) V_a} \qquad (64)$$

where $\mathbf{T_{da}}$ is the (N_d x N_a) matrix transformation between active to dependent degrees of freedom.

Now, the total global displacement vector can be written as:

$$\mathbf{V} = \begin{Bmatrix} \mathbf{V_a} \\ \mathbf{V_d} \end{Bmatrix} = \begin{Bmatrix} \mathbf{I_{aa}} \\ \mathbf{T_{da}} \end{Bmatrix} \mathbf{V}_a = \overline{\mathbf{T}} \mathbf{V}_a \qquad (65)$$

where $\mathbf{I_{aa}}$ is a (N_a x N_a) <u>unit</u> matrix.

Substitution of (64) into the EOM (61) and pre-multiplying by $\overline{\mathbf{T}}^{\mathbf{T}}$ gives:

$$\overline{\mathbf{T}}^T \mathbf{M}\left(\overline{\mathbf{T}} \ddot{\mathbf{V}}_\mathbf{a}\right) + \overline{\mathbf{T}}^T \mathbf{K}\left(\overline{\mathbf{T}} \mathbf{V_a}\right) = \overline{\mathbf{T}}^T \mathbf{F} + \overline{\mathbf{T}}^T \mathbf{Q}$$

or

$$\mathbf{M_a \ddot{V}_a + K_a V_a = F_a + Q_a} \qquad (66)$$

where the mass and stiffness matrices are reduced to (N_a x N_a) active DOF. Note that,

$$\mathbf{M_a} = \overline{\mathbf{T}}^T \mathbf{M} \overline{\mathbf{T}}, \; \mathbf{K_a} = \overline{\mathbf{T}}^T \mathbf{K} \overline{\mathbf{T}}, \mathbf{F_a} = \overline{\mathbf{T}}^T \mathbf{F}, \; \mathbf{Q_a} = \overline{\mathbf{T}}^T \mathbf{Q} \qquad (67)$$

The system of equations (66) accounts only for the active degrees of freedom N_a.

EXAMPLE: Using ONE finite element determine the first natural frequency for the beam configurations (pin-pin ends, fixed end-free end, fixed end-fixed end) and compare the results with avialable closed form formulae. Show the percentage error. The beam has length L, cross sectional area A, inertia Ip and elastic modulus E.

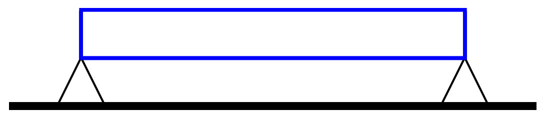
The generalized mass and stiffness matrices for beam bending are:

$$Me = \frac{\gamma \cdot L}{420}\cdot\begin{pmatrix}156 & 22\cdot L & 54 & -13\cdot L\\ 22\cdot L & 4\cdot L^2 & 13\cdot L & -3\cdot L^2\\ 54 & 13\cdot L & 156 & -22\cdot L\\ -13\cdot L & -3\cdot L^2 & -22\cdot L & 4\cdot L^2\end{pmatrix} \qquad Ke = \frac{E\cdot I}{L^3}\cdot\begin{pmatrix}12 & 6\cdot L & -12 & 6\cdot L\\ 6\cdot L & 4\cdot L^2 & -6\cdot L & 2\cdot L^2\\ -12 & -6\cdot L & 12 & -6\cdot L\\ 6\cdot L & 2\cdot L^2 & -6\cdot L & 4\cdot L^2\end{pmatrix} \qquad \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

displacement & slope (left end)

displacement & slope (right end)

Pin-Pin Ends: No displacement and moment at each end



set boundary conditions:

$\psi_1 := 0 \qquad \psi_3 := 0 \qquad Moment_1 := 0 \qquad Moment_2 := 0$

$$-\omega^2\cdot\left(\frac{\gamma\cdot L}{420}\right)\cdot\begin{pmatrix}156 & 22\cdot L & 54 & -13\cdot L\\ 22\cdot L & 4\cdot L^2 & 13\cdot L & -3\cdot L^2\\ 54 & 13\cdot L & 156 & -22\cdot L\\ -13\cdot L & -3\cdot L^2 & -22\cdot L & 4\cdot L^2\end{pmatrix}\cdot\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} + \frac{E\cdot Ip}{L^3}\cdot\begin{pmatrix}12 & 6\cdot L & -12 & 6\cdot L\\ 6\cdot L & 4\cdot L^2 & -6\cdot L & 2\cdot L^2\\ -12 & -6\cdot L & 12 & -6\cdot L\\ 6\cdot L & 2\cdot L^2 & -6\cdot L & 4\cdot L^2\end{pmatrix}\cdot\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = \begin{pmatrix}Force_1\\ Moment_1\\ Force_2\\ Moment_2\end{pmatrix}$$

$$-\omega^2\cdot\left(\frac{\gamma\cdot L}{420}\right)\cdot\begin{pmatrix}156 & 22\cdot L & 54 & -13\cdot L\\ 22\cdot L & 4\cdot L^2 & 13\cdot L & -3\cdot L^2\\ 54 & 13\cdot L & 156 & -22\cdot L\\ -13\cdot L & -3\cdot L^2 & -22\cdot L & 4\cdot L^2\end{pmatrix}\cdot\begin{pmatrix}0\\ \psi_2\\ 0\\ \psi_4\end{pmatrix} + \frac{E\cdot Ip}{L^3}\cdot\begin{pmatrix}12 & 6\cdot L & -12 & 6\cdot L\\ 6\cdot L & 4\cdot L^2 & -6\cdot L & 2\cdot L^2\\ -12 & -6\cdot L & 12 & -6\cdot L\\ 6\cdot L & 2\cdot L^2 & -6\cdot L & 4\cdot L^2\end{pmatrix}\cdot\begin{pmatrix}0\\ \psi_2\\ 0\\ \psi_4\end{pmatrix} = \begin{pmatrix}Force_1\\ 0\\ Force_2\\ 0\end{pmatrix}$$

which reduces to a 2 by 2 system of equations for free vibration

$$-\omega^2\cdot\left(\frac{\gamma\cdot L}{420}\right)\cdot\begin{pmatrix}4\cdot L^2 & -3\cdot L^2\\ -3\cdot L^2 & 4\cdot L^2\end{pmatrix}\cdot\begin{pmatrix}\psi_2\\ \psi_4\end{pmatrix} + \frac{E\cdot Ip}{L^3}\cdot\begin{pmatrix}4\cdot L^2 & 2\cdot L^2\\ 2\cdot L^2 & 4\cdot L^2\end{pmatrix}\cdot\begin{pmatrix}\psi_2\\ \psi_4\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

the natural frequencies can be determined by the eigenvalue analysis

ORIGIN := 1

$$-\omega^2\cdot I + M^{-1}\cdot K = 0$$

$$M^{-1}\cdot K = \left(\frac{420}{\gamma\cdot L}\right)\cdot\frac{1}{\left(7\cdot L^4\right)}\cdot\begin{pmatrix}4\cdot L^2 & 3\cdot L^2\\ 3\cdot L^2 & 4\cdot L^2\end{pmatrix}\cdot\left[\frac{E\cdot Ip}{L^3}\cdot\begin{pmatrix}4\cdot L^2 & 2\cdot L^2\\ 2\cdot L^2 & 4\cdot L^2\end{pmatrix}\right] \rightarrow \begin{pmatrix}\frac{1320}{L^4\cdot\gamma}\cdot E\cdot Ip & \frac{1200}{L^4\cdot\gamma}\cdot E\cdot Ip\\ \frac{1200}{L^4\cdot\gamma}\cdot E\cdot Ip & \frac{1320}{L^4\cdot\gamma}\cdot E\cdot Ip\end{pmatrix}$$

$$A := \begin{pmatrix}1320 & 1200\\ 1200 & 1320\end{pmatrix}$$

$\lambda := \mathrm{sort}(\mathrm{eigenvals}(A)) \qquad i := 1..2 \qquad \omega_i := \sqrt{\lambda_i} \qquad \omega = \begin{pmatrix}10.954\\ 50.2\end{pmatrix}$

$$\omega_1 = \frac{10.954}{L^2}\cdot\sqrt{\frac{E\cdot Ip}{\gamma}}$$

$$\omega_1 = \frac{1^2\cdot\pi^2}{L^2}\cdot\sqrt{\frac{E\cdot Ip}{\gamma}}$$

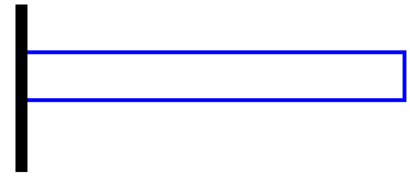
exact first natural frequency

$$error := \left|\frac{10.954-\pi^2}{\pi^2}\right|\cdot 100$$

$error = 10.987$ %

Fixed end-Free end: zero displacement and slope at the fixed end, and zero shear and moment at the free end.

set boundary conditions:



$\psi_1 := 0 \qquad \psi_2 := 0 \qquad Force_2 := 0 \qquad Moment_2 := 0$

$$-\omega^2 \cdot \left(\frac{\gamma \cdot L}{420}\right) \cdot \begin{pmatrix} 156 & 22 \cdot L & 54 & -13 \cdot L \\ 22 \cdot L & 4 \cdot L^2 & 13 \cdot L & -3 \cdot L^2 \\ 54 & 13 \cdot L & 156 & -22 \cdot L \\ -13 \cdot L & -3 \cdot L^2 & -22 \cdot L & 4 \cdot L^2 \end{pmatrix} \cdot \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} + \frac{E \cdot Ip}{L^3} \cdot \begin{pmatrix} 12 & 6 \cdot L & -12 & 6 \cdot L \\ 6 \cdot L & 4 \cdot L^2 & -6 \cdot L & 2 \cdot L^2 \\ -12 & -6 \cdot L & 12 & -6 \cdot L \\ 6 \cdot L & 2 \cdot L^2 & -6 \cdot L & 4 \cdot L^2 \end{pmatrix} \cdot \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} Force_1 \\ Moment_1 \\ Force_2 \\ Moment_2 \end{pmatrix}$$

$$-\omega^2 \cdot \left(\frac{\gamma \cdot L}{420}\right) \cdot \begin{pmatrix} 156 & 22 \cdot L & 54 & -13 \cdot L \\ 22 \cdot L & 4 \cdot L^2 & 13 \cdot L & -3 \cdot L^2 \\ 54 & 13 \cdot L & 156 & -22 \cdot L \\ -13 \cdot L & -3 \cdot L^2 & -22 \cdot L & 4 \cdot L^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ \psi_4 \end{pmatrix} + \frac{E \cdot Ip}{L^3} \cdot \begin{pmatrix} 12 & 6 \cdot L & -12 & 6 \cdot L \\ 6 \cdot L & 4 \cdot L^2 & -6 \cdot L & 2 \cdot L^2 \\ -12 & -6 \cdot L & 12 & -6 \cdot L \\ 6 \cdot L & 2 \cdot L^2 & -6 \cdot L & 4 \cdot L^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} Force_1 \\ Moment_1 \\ 0 \\ 0 \end{pmatrix}$$

which reduces to a 2 by 2 system of equations for free vibration

$$-\omega^2 \cdot \left(\frac{\gamma \cdot L}{420}\right) \cdot \begin{pmatrix} 156 & -22 \cdot L \\ -22 \cdot L & 4 \cdot L^2 \end{pmatrix} \cdot \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} + \frac{E \cdot Ip}{L^3} \cdot \begin{pmatrix} 12 & -6 \cdot L \\ -6 \cdot L & 4 \cdot L^2 \end{pmatrix} \cdot \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now the natural frequencies can be determined by the eigenvalue analysis

$$-\omega^2 \cdot I + M^{-1} \cdot K = 0$$

$$M^{-1} \cdot K = \blacksquare \qquad \frac{420}{\gamma \cdot L} \cdot \left[\frac{1}{\left(140 \cdot L^2\right)} \cdot \begin{pmatrix} 4 \cdot L^2 & 22 \cdot L \\ 22 \cdot L & 156 \end{pmatrix}\right] \cdot \left[\frac{E \cdot I}{L^3} \cdot \begin{pmatrix} 12 & -6 \cdot L \\ -6 \cdot L & 4 \cdot L^2 \end{pmatrix}\right] \rightarrow \begin{pmatrix} \frac{-252}{\gamma \cdot L^4} \cdot E \cdot I & \frac{192}{\gamma \cdot L^3} \cdot E \cdot I \\ \frac{-2016}{L^5 \cdot \gamma} \cdot E \cdot I & \frac{1476}{\gamma \cdot L^4} \cdot E \cdot I \end{pmatrix}$$

$$A := \begin{pmatrix} -252 & 192 \\ -2016 & 1476 \end{pmatrix}$$

$i := 1 .. 2$

$\lambda := sort(eigenvals(A))$

$\omega_i := \sqrt{\lambda_i} \qquad \omega = \begin{pmatrix} 3.533 \\ 34.807 \end{pmatrix} \qquad \omega_1 = \frac{3.533}{L^2} \cdot \sqrt{\frac{E \cdot Ip}{\gamma}}$

exact first natural frequency

$$\omega_1 = \frac{1.875104^2}{L^2} \cdot \sqrt{\frac{E \cdot Ip}{\gamma}}$$

$1.875104^2 = 3.516$

$$error := \left| \frac{3.533 - 1.875104^2}{1.875104^2} \right| \cdot 100 \qquad error = 0.483 \ \%$$

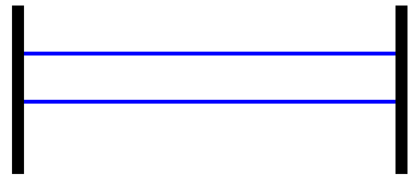
Fixed End - Fixed End: Reduce beam into two parts, each of length L/2. The boundaty conditions become zero displacement and slope at the fixed end, and zero slope at the other end of the 1/2 beam.

$L = \frac{L_o}{2}$ where Lo is the original length of the beam

set boundary conditions:

$\psi_1 := 0 \qquad \psi_2 := 0 \qquad \psi_4 := 0 \qquad Force_2 := 0$

$$-\omega^2 \cdot \left(\frac{\gamma \cdot L}{420}\right) \cdot \begin{pmatrix} 156 & 22 \cdot L & 54 & -13 \cdot L \\ 22 \cdot L & 4 \cdot L^2 & 13 \cdot L & -3 \cdot L^2 \\ 54 & 13 \cdot L & 156 & -22 \cdot L \\ -13 \cdot L & -3 \cdot L^2 & -22 \cdot L & 4 \cdot L^2 \end{pmatrix} \cdot \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} + \frac{E \cdot Ip}{L^3} \cdot \begin{pmatrix} 12 & 6 \cdot L & -12 & 6 \cdot L \\ 6 \cdot L & 4 \cdot L^2 & -6 \cdot L & 2 \cdot L^2 \\ -12 & -6 \cdot L & 12 & -6 \cdot L \\ 6 \cdot L & 2 \cdot L^2 & -6 \cdot L & 4 \cdot L^2 \end{pmatrix} \cdot \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} Force_1 \\ Moment_1 \\ Force_2 \\ Moment_2 \end{pmatrix}$$



$$-\omega^2 \cdot \left(\frac{\gamma \cdot L}{420}\right) \cdot \begin{pmatrix} 156 & 22 \cdot L & 54 & -13 \cdot L \\ 22 \cdot L & 4 \cdot L^2 & 13 \cdot L & -3 \cdot L^2 \\ 54 & 13 \cdot L & 156 & -22 \cdot L \\ -13 \cdot L & -3 \cdot L^2 & -22 \cdot L & 4 \cdot L^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ 0 \end{pmatrix} + \frac{E \cdot Ip}{L^3} \cdot \begin{pmatrix} 12 & 6 \cdot L & -12 & 6 \cdot L \\ 6 \cdot L & 4 \cdot L^2 & -6 \cdot L & 2 \cdot L^2 \\ -12 & -6 \cdot L & 12 & -6 \cdot L \\ 6 \cdot L & 2 \cdot L^2 & -6 \cdot L & 4 \cdot L^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ 0 \end{pmatrix} = \begin{pmatrix} Force_1 \\ Moment_1 \\ 0 \\ Moment_2 \end{pmatrix}$$

which reduces to just one equation for free vibration of the deflection at midspan

$$-\omega^2 \cdot \left(\frac{\gamma \cdot L}{420}\right) \cdot 156 \cdot \psi_3 + \frac{E \cdot Ip}{L^3} \cdot 12 \cdot \psi_3 = 0$$

Now the natural frequency is $\omega = \sqrt{\frac{420 \cdot 12 \cdot 2^4}{\left[\left[\left(L_o\right)^4 \cdot \gamma\right]\right] \cdot 156} \cdot E \cdot Ip}$

$$\omega_1 = \frac{22.736}{\left(L_o\right)^2} \cdot \sqrt{\frac{E \cdot Ip}{\gamma}}$$

$$\omega_1 = \frac{4.730041^2}{\left(L_o\right)^2} \cdot \sqrt{\frac{E \cdot Ip}{\gamma}}$$ exact first natural frequency

$$error := \left|\frac{22.736 - 4.730041^2}{4.730041^2}\right| \cdot 100$$

$error = 1.621$ %