Handout # 13 (MEEN 617)

Numerical Integration to Find Time Response of MDOF mechanical system

The EOMS for a linear mechanical system are

$$\mathbf{M}\,\ddot{\mathbf{U}}+\mathbf{D}\,\dot{\mathbf{U}}+\mathbf{K}\,\mathbf{U}_{(t)}=\mathbf{F}_{(t)} \qquad (1)$$

where $\mathbf{U}, \dot{\mathbf{U}},$ and $\ddot{\mathbf{U}}$ are the **vectors** of generalized displacement, velocity and acceleration, respectively; and $\mathbf{F}_{(t)}$ is the vector of generalized (external forces) acting on the system.
$\mathbf{M}, \mathbf{D}, \mathbf{K}$ represent the matrices of inertia, viscous damping and stiffness coefficients, respectively[1]. The solution of Eq. (1) is unique with specified initial conditions $\mathbf{U_o}\ \ ,\dot{\mathbf{U}}_\mathbf{o}$.

Numerical solution methods

- **direct numerical integration**
- **modal analysis (mode displacement or mode acceleration)**

(Undamped and Damped) Mode superposition methods are discussed earlier, see Lecture notes 7,8 and 11. The accuracy of these methods is determined by the number of modes selected. It is also important the time spent in performing the eigenvector analysis. **Modal superposition methods are most general. elegant, and very powerful.**

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[1] The matrices are square with *n*-rows = *n* columns, while the vectors are n-rows.

Direct numerical integration implies a marching in time, step-by-step procedure, solving the algebraic form of Eq. (1) at specific time values. The ***direct*** qualification implies that, prior to the numerical integration; no transformation of the EOM (1) is carried out.

In a numerical integration method, EOM (1) is satisfied at discrete time intervals, Δt apart. In essence, a sort of **quasi-*static*** equilibrium is sought at each time step.

Recall that a **reliable numerical integration** scheme should

a) reproduce EOM as time step $\Delta t \rightarrow 0$
b) provide, as with physical model, bounded solutions for any size of time step, i.e. method should be stable
c) reproduce the physical response with fidelity and accuracy.

Numerical integration methods are usually divided into two categories, implicit and explicit.

Consider the ODEs
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{2}$$

In an **explicit** numerical scheme, the ODEs are represented in terms of known values at a prior time step, i.e.

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \Delta t\, \mathbf{f}(\mathbf{x}_i, t_i), \tag{3}$$

while in an **implicit numerical** scheme
$$\mathbf{x}_{i+1} = \mathbf{x}_i + \Delta t\, \mathbf{f}(\mathbf{x}_{i+1}, t_i) \tag{4}$$

Explicit numerical schemes are **conditionally stable**. That is, they provide bounded numerical solutions only for (very) small time steps. For example,

$$\Delta t \le \tau_{crit} = \frac{T_n}{\pi} \qquad (5)$$

where $T_n = \dfrac{2\pi}{\omega_n}$ and $\omega_n = \sqrt{K/M}$ are the natural period and natural frequency of the system, respectively. The restriction on the time step is too severe when analyzing **stiff systems**, i.e. those with large natural frequencies.

Note that in a MDOF system, the condition defined in Eq. (5) is too restrictive since T_n refers to the smallest period of natural motion (largest natural frequency). More often than not, this critical time is NOT known unless the eigenanalysis is completed. In nonlinear systems, the critical time step limit worsens.

Implicit numerical schemes are **unconditionally stable**, i.e. do not impose a restriction on the size of the time step Δt. (However, accuracy may be compromised if too large time steps are used).

An integration method is ***unconditionally stable*** *if the solution for any initial condition does not grow without bound for any time step* Δt*, in particular when is large* $\Delta t / T_n$. *The method is only conditionally stable if the above holds provided that* $\Delta t / T_n$ *is smaller than a certain value, usually called the* **stability limit.**

The Wilson-θ method[2]

It is an extension of the **linear-acceleration method**. That is, within a time step, as shown in Figure 1, the acceleration vector is proportional to time. Over the time interval $0 \le \tau \le \theta\,\Delta t$,

$$\ddot{\mathbf{U}}_{t+\tau} = \ddot{\mathbf{U}}_t + \frac{\tau}{\theta\,\Delta t}\left[\ddot{\mathbf{U}}_{t+\theta\Delta t} - \ddot{\mathbf{U}}_t\right] \qquad (6)$$

with $\theta \ge 1$.

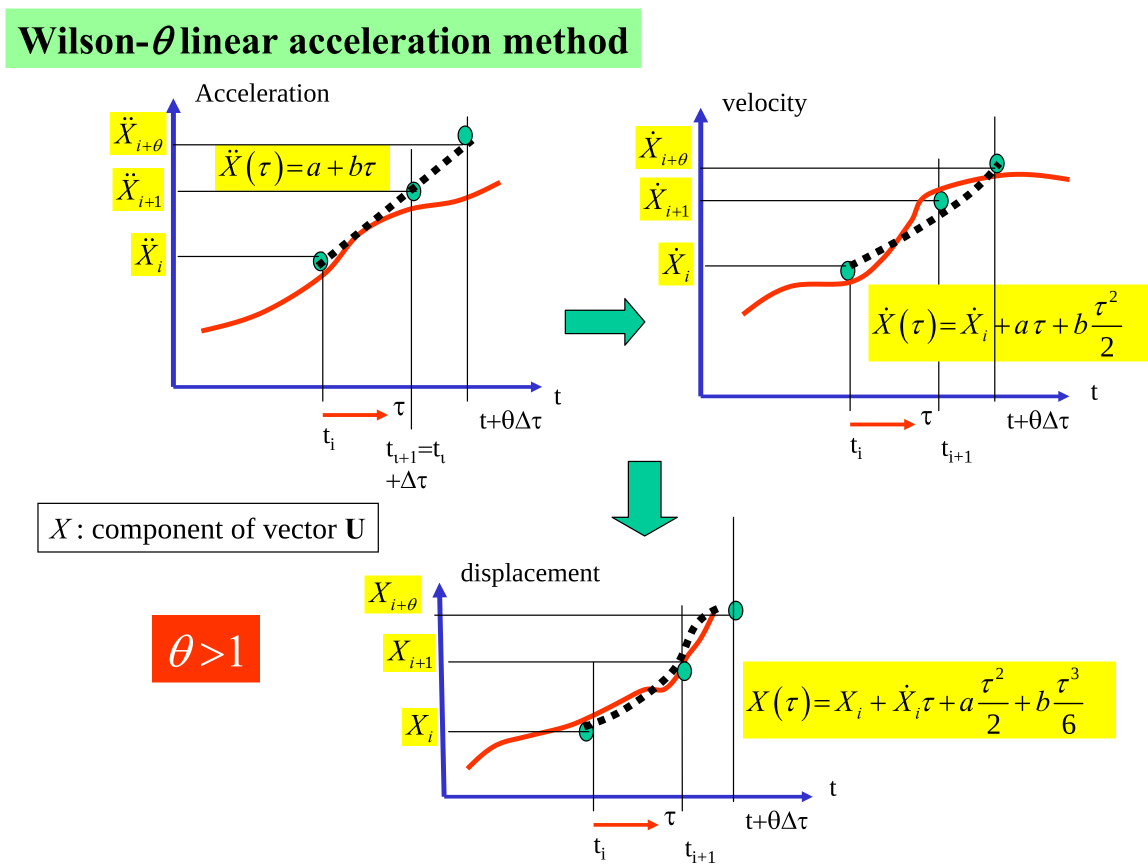

Fig 1. Depiction of components of acceleration, velocity and displacement for numerical integration – Wilson-θ method

Integration of Eq. (6) gives the vector of velocity as

[2] This numerical method is extremely popular among the structural dynamics community.

$$\dot{\mathbf{U}}_{t+\tau} = \dot{\mathbf{U}}_t + \tau\,\ddot{\mathbf{U}}_t + \frac{\tau^2}{2\theta\,\Delta t}\left[\ddot{\mathbf{U}}_{t+\theta\Delta t} - \ddot{\mathbf{U}}_t\right] \qquad (7)$$

One more integration delivers the vector of displacements as

$$\mathbf{U}_{t+\tau} = \mathbf{U}_t + \tau\dot{\mathbf{U}}_t + \frac{\tau^2}{2}\ddot{\mathbf{U}}_t + \frac{\tau^3}{6\theta\,\Delta t}\left[\ddot{\mathbf{U}}_{t+\theta\Delta t} - \ddot{\mathbf{U}}_t\right] \qquad (8)$$

Specification of the vectors at time $\tau = \theta\,\Delta t$

$$\begin{aligned}\dot{\mathbf{U}}_{t+\theta\Delta t} &= \dot{\mathbf{U}}_t + \theta\Delta t\,\ddot{\mathbf{U}}_t + \frac{(\theta\Delta t)^2}{2\theta\,\Delta t}\left[\ddot{\mathbf{U}}_{t+\theta\Delta t} - \ddot{\mathbf{U}}_t\right] \\ &= \dot{\mathbf{U}}_t + \theta\Delta t\left[\ddot{\mathbf{U}}_{t+\theta\Delta t} + \ddot{\mathbf{U}}_t\right]\end{aligned} \qquad (9)$$

$$\begin{aligned}\mathbf{U}_{t+\theta\Delta t} &= \mathbf{U}_t + \theta\Delta t\,\dot{\mathbf{U}}_t + \frac{(\theta\Delta t)^2}{2}\ddot{\mathbf{U}}_t + \frac{(\theta\Delta t)^3}{6\theta\,\Delta t}\left[\ddot{\mathbf{U}}_{t+\theta\Delta t} - \ddot{\mathbf{U}}_t\right] \\ &= \mathbf{U}_t + \theta\Delta t\,\dot{\mathbf{U}}_t + \frac{(\theta\Delta t)^2}{6}\left[\ddot{\mathbf{U}}_{t+\theta\Delta t} + 2\ddot{\mathbf{U}}_t\right]\end{aligned} \qquad (10)$$

From Eqs. (9) and (10), solve for the acceleration and velocity at the end of the interval, i.e.

$$\ddot{\mathbf{U}}_{t+\theta\Delta t} = \frac{6}{(\theta\Delta t)^2}\left[\mathbf{U}_{t+\theta\Delta t} - \mathbf{U}_t\right] - \frac{6}{\theta\Delta t}\dot{\mathbf{U}}_t - 2\,\ddot{\mathbf{U}}_t \qquad (11)$$

$$\dot{\mathbf{U}}_{t+\theta\Delta t} = \frac{3}{\theta\Delta t}\left[\mathbf{U}_{t+\theta\Delta t} - \mathbf{U}_t\right] - 2\,\dot{\mathbf{U}}_t - \frac{\theta\Delta t}{2}\ddot{\mathbf{U}}_t \qquad (12)$$

Now, state EOM (1) at the end of the time interval $(t + \theta\,\Delta t)$

$$\text{at } t + \theta\,\Delta t \Rightarrow \mathbf{M}\,\ddot{\mathbf{U}}_{t+\theta\,\Delta t} + \mathbf{D}\,\dot{\mathbf{U}}_{t+\theta\,\Delta t} + \mathbf{K}\,\mathbf{U}_{t+\theta\,\Delta t} = \overline{\mathbf{F}}_{t+\theta\,\Delta t} \qquad (13)$$

where

$$\overline{\mathbf{F}}_{t+\theta\,\Delta t} = \mathbf{F}_t + \theta\left[\mathbf{F}_{t+\Delta t} - \mathbf{F}_t\right] \qquad (14)$$

is a (time) projection of the external force vector.

Substitute Eqs. (11) and (12) into (13) to determine an equation from which $\mathbf{U}_{t+\theta\Delta t}$ can be obtained.

Next, substitute this vector into Eq (12) and (11) to obtain $\dot{\mathbf{U}}_{t+\theta\Delta t}$ and $\ddot{\mathbf{U}}_{t+\theta\Delta t}$.

Lastly, use these vectors to determine in Eqs. (8) and (9) the displacement and velocity vectors at $\tau=\Delta t$, i.e. $\dot{\mathbf{U}}_{t+\Delta t}$ and $\mathbf{U}_{t+\Delta t}$.

The **resulting system of equations** for **solution at each time step** is

$$\hat{\mathbf{K}}\,\mathbf{U}_{t+\theta\Delta t}=\hat{\mathbf{F}}_{t+\theta\Delta t} \qquad (15)$$

where

$$\hat{\mathbf{K}}=\mathbf{K}+a_o\,\mathbf{M}+a_1\mathbf{D} \qquad (16)$$

is the ***effective* stiffness matrix**, and

$$\begin{aligned}\hat{\mathbf{F}}_{t+\theta\Delta t}=\mathbf{F}_t+\theta\left[\mathbf{F}_{t+\Delta t}-\mathbf{F}_t\right]+&\\ \mathbf{M}\left(a_o\,\mathbf{U}_t+a_2\dot{\mathbf{U}}_t+2\,\ddot{\mathbf{U}}_t\right)+&\\ \mathbf{D}\left(a_1\,\mathbf{U}_t+2\,\dot{\mathbf{U}}_t+a_3\,\ddot{\mathbf{U}}_t\right)&\end{aligned} \qquad (17)$$

is the ***effective load vector*** *at* $t+\theta\Delta t$. Above,

$$a_o = \frac{6}{(\theta\,\Delta t)^2},\ a_1 = \frac{3}{\theta\,\Delta t},\ a_2 = 2\,a_1, a_3 = \frac{\theta\,\Delta t}{2}, a_4 = \frac{a_o}{\theta},$$
$$a_5 = -\frac{a_2}{\theta},\ a_6 = 1 - \frac{3}{\theta},\ a_7 = \frac{\Delta t}{2},\ a_8 = \frac{\Delta t^2}{6} \qquad (18)$$

The *effective* stiffness matrix can be easily decomposed (once) as $\hat{\mathbf{K}} = \mathbf{L}\,\mathbf{D_g}\,\mathbf{L^T}$ If **M, K, D** are symmetric.

Otherwise, $\hat{\mathbf{K}} = \mathbf{L}\,\mathbf{U}_\Delta$, i.e. the product of a lower triangular and upper triangular form matrices.

Hence, the solution of Eq. (15) proceeds step by step in a process of backward and forward substitutions, i.e.

$$\mathbf{L}\ \mathbf{Y} = \hat{\mathbf{F}}_{t+\theta\Delta t}$$
$$\mathbf{U}_\Delta\ \mathbf{U}_{t+\theta\Delta t} = \mathbf{Y} \qquad (19)$$

Solution of Eq. (19) gives $\mathbf{U}_{t+\theta\Delta t}$. Next, calculate the displacements, velocity, and acceleration vectors at $(t+\Delta t)$ using

$$\ddot{\mathbf{U}}_{t+\Delta t} = a_4\left[\mathbf{U}_{t+\theta\Delta t} - \mathbf{U}_t\right] + a_5\,\dot{\mathbf{U}}_t + a_6\,\ddot{\mathbf{U}}_t$$
$$\dot{\mathbf{U}}_{t+\Delta t} = \dot{\mathbf{U}}_t + a_7\left[\ddot{\mathbf{U}}_{t+\Delta t} + \ddot{\mathbf{U}}_t\right] \qquad (20)$$
$$\mathbf{U}_{t+\Delta t} = \mathbf{U}_t + \Delta t\,\dot{\mathbf{U}}_t + a_8\left[\ddot{\mathbf{U}}_{t+\Delta t} + 2\ddot{\mathbf{U}}_t\right]$$

Analysis shows that θ =1.37-1.40 for ***unconditional stability***.

The Newmark-β method

This method is also an extension of the **linear-acceleration method**. The velocity and displacement vectors at the end of the time interval $(t+\Delta t)$ are, as shown in Fig 2:

$$\dot{\mathbf{U}}_{t+\Delta t} = \dot{\mathbf{U}}_t + \left[(1-\delta)\ddot{\mathbf{U}}_t + \delta \ddot{\mathbf{U}}_{t+\Delta t} \right] \Delta t \backslash$$

$$\mathbf{U}_{t+\Delta t} = \mathbf{U}_t + \Delta t\, \dot{\mathbf{U}}_t + \left[(\tfrac{1}{2}-\alpha)\ddot{\mathbf{U}}_t + \alpha \ddot{\mathbf{U}}_{t+\Delta t} \right] \Delta t^2 \qquad (21)$$

where (α, δ) are parameters selected to promote numerical stability and/or gain in accuracy.

Linear acceleration method

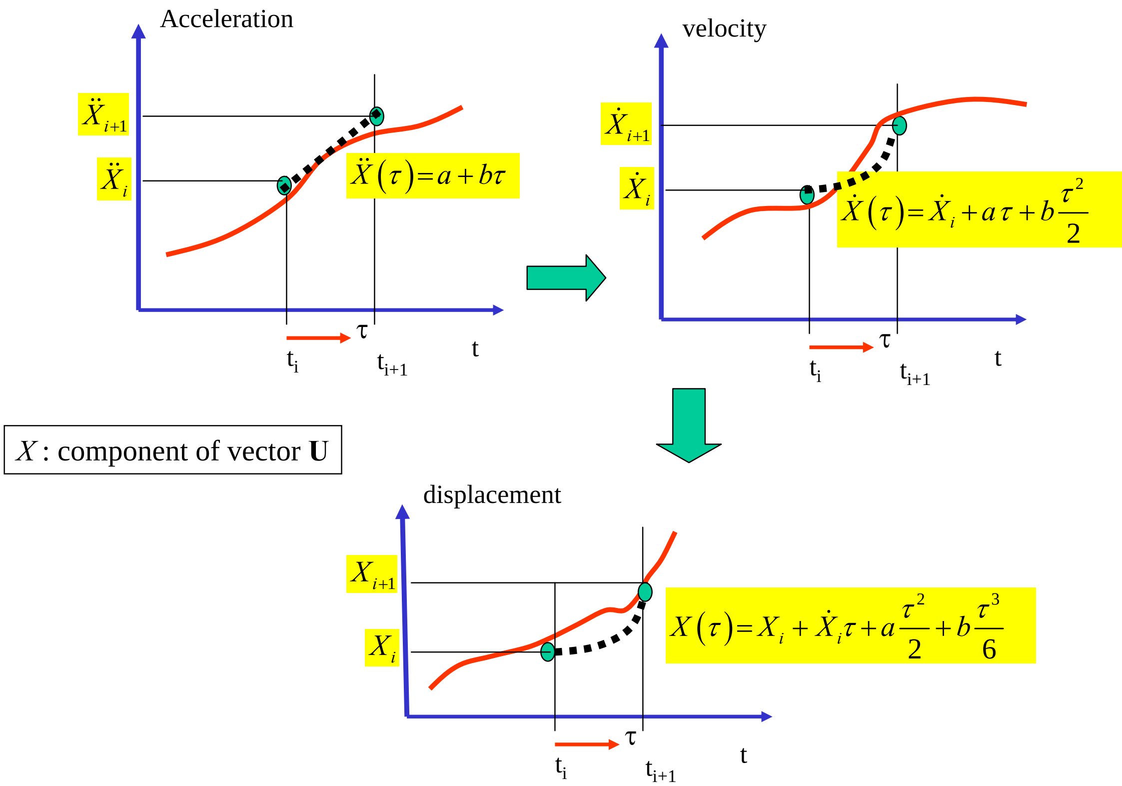

Fig 2. Depiction of components of acceleration, velocity and displacement for numerical integration – Newmark method

$\delta = \frac{1}{2}, \alpha = \frac{1}{6}$, Eq. (21) represents the linear acceleration method (θ=1 in Wilson's method).

$\delta = \frac{1}{2}, \alpha = \frac{1}{4}$, brings **unconditional stability**, and Eq. (21) represents the **average acceleration method** discussed for SDOF systems. See *Lecture Notes 6*.

The method solves the algebraic form of EOM (1) at the end of the time interval

$$t + \Delta t \Rightarrow \mathbf{M}\,\ddot{\mathbf{U}}_{t+\Delta t} + \mathbf{D}\,\dot{\mathbf{U}}_{t+\Delta t} + \mathbf{K}\,\mathbf{U}_{t+\Delta t} = \mathbf{F}_{t+\Delta t} \quad (22)$$

Substitution of Eqs. (21) above leads to the algebraic equation:

$$\hat{\mathbf{K}}\,\mathbf{U}_{t+\Delta t} = \hat{\mathbf{F}}_{t+\Delta t} \quad (23)$$

where

$$\hat{\mathbf{K}} = \mathbf{K} + a_o\,\mathbf{M} + a_1\mathbf{D} \quad (24)$$

is the ***effective* stiffness matrix**, and

$$\hat{\mathbf{F}}_{t+\Delta t} = \mathbf{F}_{t+\Delta t} + \mathbf{M}\left(a_o\,\mathbf{U}_t + a_2\dot{\mathbf{U}}_t + a_3\,\ddot{\mathbf{U}}_t\right) + \mathbf{D}\left(a_1\,\mathbf{U}_t + a_4\,\dot{\mathbf{U}}_t + a_5\,\ddot{\mathbf{U}}_t\right) \quad (25)$$

is the ***effective load vector*** *at* $t + \Delta t$. Above,

$$a_o = \frac{6}{\alpha\,\Delta t^2},\; a_1 = \frac{\delta}{\alpha\,\Delta t},\; a_2 = \frac{1}{\alpha\,\Delta t}, a_3 = \frac{1}{2\alpha} - 1, a_4 = \frac{\delta}{\alpha} - 1,$$
$$a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha} - 2\right),\; a_6 = \Delta t\left(1-\alpha\right),\; a_7 = \delta\Delta t \qquad (26)$$

In general, it is recommended that $\delta \geq \tfrac{1}{2}, \alpha \geq \tfrac{1}{4}\left(\tfrac{1}{2}+\delta\right)^2$ (27)

The *effective* stiffness matrix is easily decomposed (once) as $\hat{\mathbf{K}} = \mathbf{L}\,\mathbf{U}_\Delta$ for solution of Eq. (23), i.e. to determine $\mathbf{U}_{t+\Delta t}$.

The vectors of acceleration and velocity follow from

$$\begin{aligned} \ddot{\mathbf{U}}_{t+\Delta t} &= a_0\left[\mathbf{U}_{t+\Delta t} - \mathbf{U}_t\right] - a_2\,\dot{\mathbf{U}}_t + a_3\,\ddot{\mathbf{U}}_t \\ \dot{\mathbf{U}}_{t+\Delta t} &= \dot{\mathbf{U}}_t + a_6\ddot{\mathbf{U}}_t + a_7\ddot{\mathbf{U}}_{t+\Delta t} \end{aligned} \qquad (28)$$

Stability of Numerical Method

In general, the numerical method solves $\hat{\mathbf{K}}\,\mathbf{U}_t = \hat{\mathbf{F}}_t$ and $\hat{\mathbf{K}}\,\mathbf{U}_{t+\Delta t} = \hat{\mathbf{F}}_{t+\Delta t}$ in two consecutive time steps.

For a sufficiently small time step,

$$\hat{\mathbf{F}}_{t+\Delta t} = \hat{\mathbf{F}}_t + \frac{\partial\hat{\mathbf{F}}}{\partial t}\Delta t + \frac{\partial^2\hat{\mathbf{F}}}{\partial t^2}\frac{\Delta t^2}{2} + \ldots. \approx \hat{\mathbf{K}}\,\mathbf{U}_t + \frac{\partial\hat{\mathbf{F}}}{\partial t}\Delta t \qquad (29)$$

Hence

$$\hat{\mathbf{K}}\,\mathbf{U}_{t+\Delta t} = \hat{\mathbf{K}}\,\mathbf{U}_t + \frac{\partial\hat{\mathbf{F}}}{\partial t}\Delta t \qquad (30)$$

After some algebraic manipulations, $\mathbf{U}_{t+\Delta t} = \widehat{\mathbf{A}}\,\mathbf{U}_t + \mathbf{L}_{\mathbf{F}}$ (31)

Hence, assume $\mathbf{L_F=0}$ for stability analysis. **A** is known as the convergence matrix, Then,

$$\begin{aligned}
&\mathbf{U}_1 = \widehat{\mathbf{A}}\,\mathbf{U}_0, \\
&\mathbf{U}_2 = \widehat{\mathbf{A}}\,\mathbf{U}_1 = \widehat{\mathbf{A}}^2\,\mathbf{U}_0 \\
&\ldots\ldots \\
&\mathbf{U}_n = \widehat{\mathbf{A}}\,\mathbf{U}_{n-1} = \widehat{\mathbf{A}}^n\,\mathbf{U}_0
\end{aligned} \tag{32}$$

The numerical solution is bounded, i.e. **stable**, if the spectral radius of the convergence matrix is less than one,

$$\rho\left(\widehat{\mathbf{A}}\right) = \left|\lambda_{\max}\right| < 1 \tag{33}$$

That is, none of its eigenvalues is greater or equal to unit.

Numerical Solution for Nonlinear Systems

The EOMS for a non-linear mechanical system are

$$\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{F}_{NL_{(t)}} = \mathbf{F}_{(t)} \tag{34}$$

where $\mathbf{F}_{NL_{(t)}} = f_{NL}\left(\mathbf{U}, \dot{\mathbf{U}}, \ddot{\mathbf{U}}\right)$ is a **nonlinear** function of $\mathbf{U}, \dot{\mathbf{U}}, \text{and } \ddot{\mathbf{U}}$, i.e. the **vectors** of generalized displacement, velocity and acceleration. Specification of Eq. (34) at two times, **sufficiently close** to each other, gives:

$$\mathbf{M}\,\ddot{\mathbf{U}}_t + \mathbf{F}_{NL_{(t)}} = \mathbf{F}_{(t)} \tag{35a}$$

$$\mathbf{M}\,\ddot{\mathbf{U}}_{t+\Delta t} + \mathbf{F}_{NL_{(t+\Delta t)}} = \mathbf{F}_{(t+\Delta t)} \tag{35b}$$

Note that in (35a) all information is known. Subtracting (b) from (a) gives:

$$\mathbf{M}\,\Delta\ddot{\mathbf{U}}_t + \Delta\mathbf{F}_{NL_{(t)}} = \Delta\mathbf{F}_{(t)} \qquad (36)$$

Now, let

$$\Delta\mathbf{F}_{NL_{(\Delta t)}} = \mathbf{F}_{NL(t+\Delta t)} - \mathbf{F}_{NL(\Delta t)} \simeq \left(\frac{\partial \mathbf{F}_{NL}}{\partial \mathbf{U}}\right)_t \Delta\mathbf{U}_t + \left(\frac{\partial \mathbf{F}_{NL}}{\partial \dot{\mathbf{U}}}\right)_t \Delta\dot{\mathbf{U}}_t + \left(\frac{\partial \mathbf{F}_{NL}}{\partial \ddot{\mathbf{U}}}\right)_t \Delta\ddot{\mathbf{U}}_t \qquad (37)$$

where

$$\Delta\ddot{\mathbf{U}}_t = \left(\ddot{\mathbf{U}}_{t+\Delta t} - \ddot{\mathbf{U}}_t\right),\ \Delta\dot{\mathbf{U}}_t = \left(\dot{\mathbf{U}}_{t+\Delta t} - \dot{\mathbf{U}}_t\right), \Delta\mathbf{U}_t = \left(\mathbf{U}_{t+\Delta t} - \mathbf{U}_t\right)$$

(38)

And define the following matrices, **evaluated at each time step,**

$$\mathbf{K}_t = \left(\frac{\partial \mathbf{F}_{NL}}{\partial \mathbf{U}}\right)_t ; \mathbf{D}_t = \left(\frac{\partial \mathbf{F}_{NL}}{\partial \dot{\mathbf{U}}}\right)_t ; \mathbf{M}_t = \left(\frac{\partial \mathbf{F}_{NL}}{\partial \ddot{\mathbf{U}}}\right)_t \qquad (39)[3]$$

These matrices represent linearized stiffness (**K**), viscous damping (**D**) and inertia (**M**) coefficients. Substitution of Eq. (39) into (37) and into Eq. (36) gives:

$$\left(\mathbf{M} + \mathbf{M}_t\right)\Delta\ddot{\mathbf{U}}_t + \mathbf{D}_t\,\Delta\dot{\mathbf{U}}_t + \mathbf{K}_t\,\Delta\mathbf{U}_t = \Delta\mathbf{F}_t \qquad (40)$$

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[3] The elements of the stiffness matrix are, for example,

$$K_{a,b} = {\partial F_{NL_b}}\big/{U_a},\ a,b = 1,2,\ldots n$$

The solution of this algebraic equation proceeds in the same form as for the SDOF system, see Lecture Notes 6.

Thus, the numerical treatment is similar, except **that at each time step,** linearized stiffness, damping and inertia coefficients need be calculated.

Although the recipe is identical; however with the apparent nonlinearity, the method does not guarantee stability for (too) large time steps.

References

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Numerical Integration of MDOF Linear systems with Viscous Damping - NEWMARK METHOD

Original by Dr. Luis San Andres for MEEN 617 class / 2008

The equations of motion are: $\mathbf{M}\, d^2\mathbf{U}/dt^2 + \mathbf{C}\, d\mathbf{U}/dt + \mathbf{K}\,\mathbf{U} = \mathbf{P}(t)$ (1)

where **M,C,K** are nxn matrices of inertia, viscous damping and stiffness coefficients, and **U**, **V**=d**U**/dt, **W**=d²**U**/dt²,and are the nx1 vectors of displacements, velocity and accelerations, resp, and **P**(t) is the nx1 vector of generalized forces.
Eq. (1) is solved numerically with initial conditions, at t=0, **Uo,Vo=**d**U**/dt

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1. Define elements of inertia, stiffness, and damping matrices:

$n := 3$ # of DOF

$$M := \begin{pmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 50 \end{pmatrix}$$

kg

$$K := \begin{pmatrix} 2\cdot 10^7 & -1\cdot 10^7 & 0 \\ -1\cdot 10^7 & 2.5\cdot 10^7 & -.5\cdot 10^7 \\ 0 & -.5\cdot 10^7 & .5\cdot 10^7 \end{pmatrix}$$

N/m

$$C := \begin{pmatrix} 5000 & 0 & 0 \\ 0 & 2500 & -1000 \\ 0 & -1000 & 1000 \end{pmatrix}\cdot .0$$

N.s/m

Input vectors of initial conditions:

$$U_0 := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ **m**

$$V_0 := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ **m/s**

$F_{max} := 10000$ **N**

$T_{impulse} := 0.01$

natural frequencies

smallest natural period $T_{min} = 0.011$

Set sampling time: $\Delta t := \dfrac{T_{min}}{2}$ $\Delta t = 5.377 \times 10^{-3}$ should be less than smallest period $\dfrac{1}{f} = \begin{pmatrix} 0.026 \\ 0.016 \\ 0.011 \end{pmatrix}$

Number of time steps: $N_{times} := 2^{10}$ $N_{times} = 1.024 \times 10^3$

Pulse load

$T_{half} := 0.5\cdot T_{impulse}$

EXAMPLE

$$F(t) := \begin{cases} F_{max}\cdot\dfrac{t}{T_{half}} & \text{if } t < T_{half} \\ \left[-F_{max}\cdot\dfrac{(t - T_{impulse})}{T_{half}}\right] & \text{if } T_{half} \le t < T_{impulse} \\ 0 & \text{otherwise} \end{cases}$$

$f_1 = 61.64$

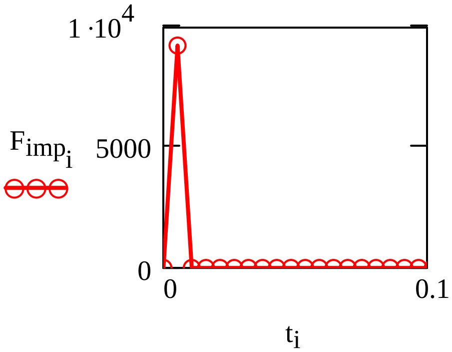
Build time and force vectors:

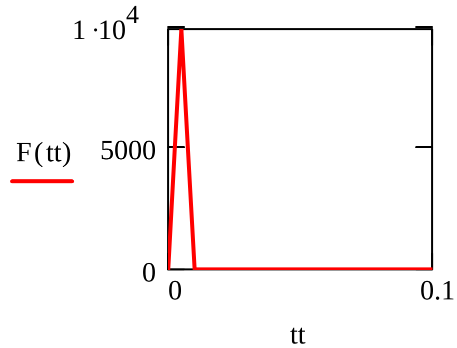
$\dfrac{T_{impulse}}{\Delta t} = 1.86$

$i := 0 .. N_{times} - 1$

$t_i := i\cdot\Delta t$ $Fimp_i := F(t_i)$ Note that impact must be well captured

Pulse load





$$\frac{impulse}{\Delta t} = 1.86$$

Numerical sampling vs. actual function

Set parameters for Newmark integration method:

$\delta := 0.5$ $\alpha := 0.25\cdot(0.5+\delta)^2$

$$a_0 := \frac{1}{\alpha\cdot\Delta t^2} \quad a_1 := \frac{\delta}{\alpha\cdot\Delta t^1} \quad a_2 := \frac{1}{\alpha\cdot\Delta t^1} \quad a_3 := \left(\frac{1}{2\cdot\alpha}\right) - 1 \quad \alpha = 0.25$$

$$a_4 := \left(\frac{\delta}{\alpha}\right) - 1 \quad a_5 := \left(\frac{\Delta t}{2}\right)\cdot\left(\frac{\delta}{\alpha} - 2\right) \quad a_6 := \Delta t\cdot(1-\delta) \quad a_7 := \delta\cdot\Delta t$$

Form effective stiffness matrix: $K_e := K + a_0\cdot M + a_1\cdot C$

Triangularize effective K matrix, since n=3 (small), best to find inverse, $B_e := K_e^{-1}$ (effective flexibility matrix)

Calculate response for impact exerted at one of DOFs

Force vector $g(t) := \begin{pmatrix} 0 \\ 0 \\ F(t) \end{pmatrix}$

numerical procedure

Find initial Acceleration (W) from EOM & include external force at t=0, i.e

initial time

$$W_0 := -M^{-1}\cdot\left(K\cdot U_0 + C\cdot V_0 - g\left(t_0\right)\right)$$

$$t_0 := 0$$

initial accel.

$$W_0^{T} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

U: displacement,
V: velocity
W: acceleration

Step by step numerical integration

$$\begin{aligned}
\mathrm{NumInt}(g) := \;& u^{\langle 0\rangle} \leftarrow U_0 \\
& v^{\langle 0\rangle} \leftarrow V_0 \\
& w^{\langle 0\rangle} \leftarrow W_0 \\
& \text{for } i \in 1 .. N_{times} - 1 \\
& \quad P \leftarrow g\left(t_i\right) \\
& \quad P \leftarrow P + M\cdot\left(a_0\cdot u^{\langle i-1\rangle} + a_2\cdot v^{\langle i-1\rangle} + a_3\cdot w^{\langle i-1\rangle}\right) + C\cdot\left(a_1\cdot u^{\langle i-1\rangle} + a_4\cdot v^{\langle i-1\rangle} + a_5\cdot w^{\langle i-1\rangle}\right) \\
& \quad u^{\langle i\rangle} \leftarrow B_e\cdot P \\
& \quad w^{\langle i\rangle} \leftarrow a_0\cdot\left(u^{\langle i\rangle} - u^{\langle i-1\rangle}\right) - a_2\cdot v^{\langle i-1\rangle} - a_3\cdot w^{\langle i-1\rangle} \\
& \quad v^{\langle i\rangle} \leftarrow v^{\langle i-1\rangle} + a_6\cdot w^{\langle i-1\rangle} + a_7\cdot w^{\langle i\rangle} \\
& \text{return } u
\end{aligned}$$

find numerical response $z := \mathrm{NumInt}(g)$ $\quad \mathrm{rows}(z) = 3 \quad \mathrm{cols}(z) = 1.024 \times 10^{3}$

Extract time responses obtained from numerical integration

$$i := 0 .. N_{times} - 1$$

$$U1_i := z_{0,i} \qquad U2_i := z_{1,i} \qquad U3_i := z_{2,i}$$

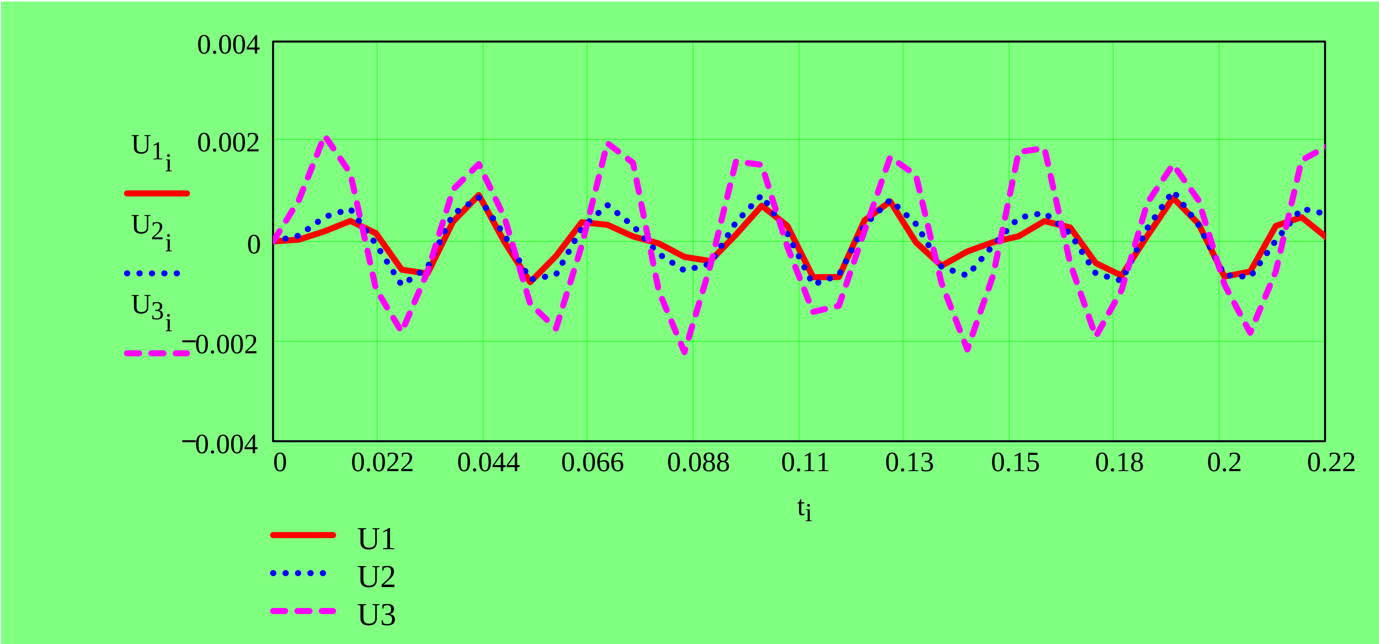
numerical procedure

RESULTS of numerical response: $\quad T_{impulse} = 0.01 \qquad T_{max\_plot} := .22$

UNDAMPED NATURAL FREQS.

$$T = \begin{pmatrix} 0.026 \\ 0.016 \\ 0.011 \end{pmatrix} \qquad f = \begin{pmatrix} 38.52 \\ 61.64 \\ 92.996 \end{pmatrix}$$

=============================== VERY LARGE TIME STEP

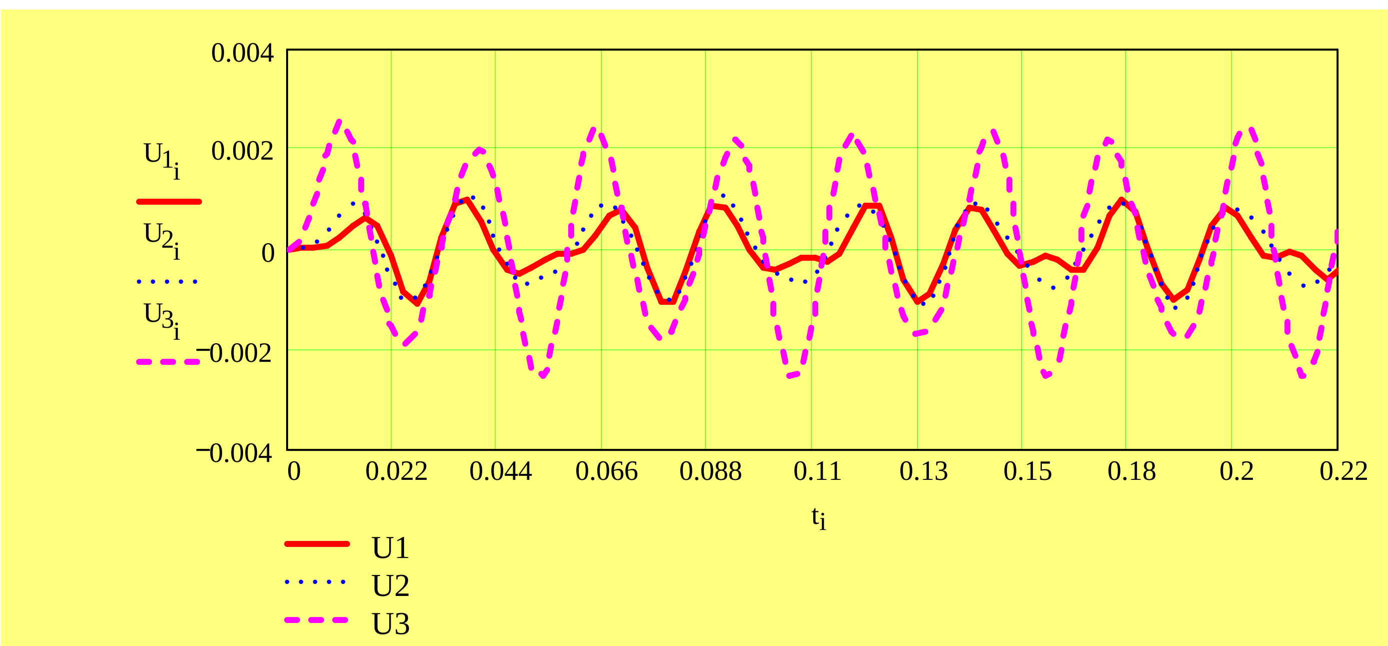


$\max(t) = 5.5$

$\Delta t = 5.377 \times 10^{-3}$

$\frac{T_{impulse}}{\Delta t} = 1.86$

=============================== LARGE TIME STEP



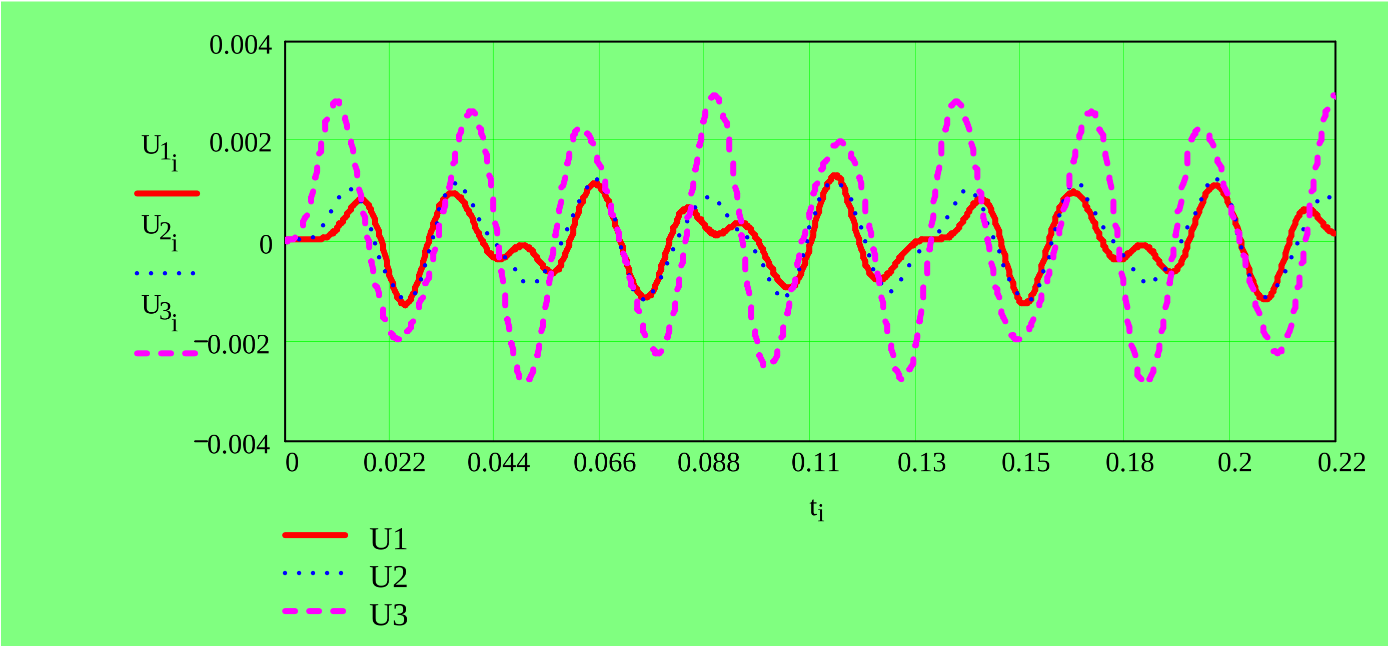
$t_{last(t)} = 2.75$

$\max(t) = 2.75$

$\Delta t = 2.688 \times 10^{-3}$

$\frac{T_{impulse}}{\Delta t} = 3.72$

============================== **SMALL TIME STEP**



$\max(t) = 0.275$

$\Delta t = 2.688 \times 10^{-4}$

$\frac{T_{impulse}}{\Delta t} = 37.199$