

MEEN 617 Handout 3

A Super Brief Introduction to the Discrete Fourier Transform

Original from Dr. Joe-Yong Kim (ME 459/659), modified by Dr. Luis San Andrés (MEEN 617, Jan 2013).

This handout is a pale substitution for the detailed and elegant discussion presented in your textbook.

The Discrete Fourier Transform

The **Fourier Transform (FT)** and its **inverse FT** are defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\omega t} dt, \quad (1)$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i2\pi\omega t} d\omega \quad (2)$$

Above note the integrals are evaluated over infinite limits.

Consider the set $\{x_n\}_{n=0,1,\dots,N-1}$ recorded at discrete times

$\{t_n\} = t_0, t_1 = t_0 + \Delta t, t_2 = t_0 + 2\Delta t, \dots, t_{N-1} = t_0 + \Delta t(N-1)$, where N is the number of samples acquired.

The **Discrete Fourier Transform (DFT)** of a spatially or **time sampled** series x_n is

$$X_m = \sum_{n=0}^{N-1} x_n e^{-i2\pi \frac{mn}{N}}, \quad m = 0, \dots, N-1. \quad (3)$$

and the **inverse DFT** is

$$x_n = \frac{1}{N} \sum_{m=0}^{N-1} X_m e^{i2\pi \frac{mn}{N}}, \quad n = 0, \dots, N-1. \quad (4)$$

Note the DFT and its inverse are the **discrete form of a truncated FT**.

The vector $\{X_m\}_{m=0,\dots,N-1} = (a_m + i b_m)$ is **complex**.

Presently, the DFT and inverse DFT can be calculated fast and efficiently by using various Fast Fourier Transform (FFT) algorithms. (e.g., the “fft” command in Matlab® or MATCAD®)

The **DFT** shows that, $X_0 = X_{N-1}^*$, $X_1 = X_{N-2}^*$, ..., $X_2 = X_{N-3}^*$, ... where (*) denotes the complex conjugate, $(a_m - i b_m)$.

In practice, software usually delivers a vector of $\frac{1}{2}N$ values (shifted), i.e.,

$$\hat{X}_0 = X_{(N-1)-k}, \hat{X}_1 = X_{(N-1)-k+1}; \dots; \hat{X}_{k-1} = X_{N-2}; \hat{X}_k = X_{N-1}; \quad k = \frac{N}{2} \quad (5)$$

The **maximum frequency** (f_{max}) of the DFT of a time series $\{x_n\}_{n=0, N-1}$ sampled at Δt satisfies the Nyquist Sampling Theorem, i.e.,

$$f_{max} \leq \frac{f_{sample}}{2} = \frac{1}{2\Delta t} \quad (6)$$

There are $k = \frac{1}{2}N$ data points in the frequency spectrum (complex numbers). Since the maximum frequency is $f_{max} = f_{sample}/2$, the frequency resolution (Δf) equals

$$\Delta f = \frac{f_{sample}}{N} = \frac{1}{N\Delta t} = \frac{1}{T} = \frac{1}{\text{time record length}} \quad (7)$$

Example 1

Figure 1(a) below shows $x_{(t)} = 1 \sin(\omega t)$, with $\omega = 2\pi f$, $f = 22 \text{ Hz}$, sampled at 100 Hz (samples/s) or $\Delta t = 0.01 \text{ s}$, and the number of points is $N = 256$ ($T_{max} = 2.55 \text{ s}$). Note that $\Delta t \ll 0.045 \text{ s}$, the period of the $f = 22 \text{ Hz}$ wave.

Figure 1(b) shows the amplitude of the DFT, $|X_m|_{m=0, \dots, N/2-1}$ versus frequency. The maximum frequency in the DFT is $f_{max} = 50 \text{ Hz}$ with a step of $\Delta f = \frac{1}{\Delta t N} = 0.391 \text{ Hz}$. The number of frequencies in the DFT is 128. Note the amplitude of the DFT $|X_m|$ shows components at other frequencies than 22 Hz.

The DFT is a collection of $k = \frac{1}{2}N$ complex numbers, i.e., it is a discrete set (not continuous). Figure 1(c) graphs the real and imaginary parts of the DFT X_m .

| | | | |
|-------------|----|---|---|
| $f_{req} =$ | 1 | * | 1 |
| | 1 | | 4.921·10 ⁻³ |
| | 2 | | 9.843·10 ⁻³ +1.683i·10 ⁻⁴ |
| | 3 | | 9.848·10 ⁻³ +3.368i·10 ⁻⁴ |
| | 4 | | 9.856·10 ⁻³ +5.059i·10 ⁻⁴ |
| | 5 | | 9.867·10 ⁻³ +6.758i·10 ⁻⁴ |
| | 6 | | 9.882·10 ⁻³ +8.468i·10 ⁻⁴ |
| | 7 | | 9.9·10 ⁻³ +1.019i·10 ⁻³ |
| | 8 | | 9.921·10 ⁻³ +1.193i·10 ⁻³ |
| | 9 | | 9.945·10 ⁻³ +1.369i·10 ⁻³ |
| | 10 | | 9.974·10 ⁻³ +1.548i·10 ⁻³ |
| | 11 | | 0.01+1.729i·10 ⁻³ |
| | 12 | | 0.01+1.913i·10 ⁻³ |
| | 13 | | 0.01+2.101i·10 ⁻³ |
| | 14 | | 0.01+2.292i·10 ⁻³ |
| | 15 | | 0.01+2.488i·10 ⁻³ |
| | 16 | | 0.01+2.688i·10 ⁻³ |

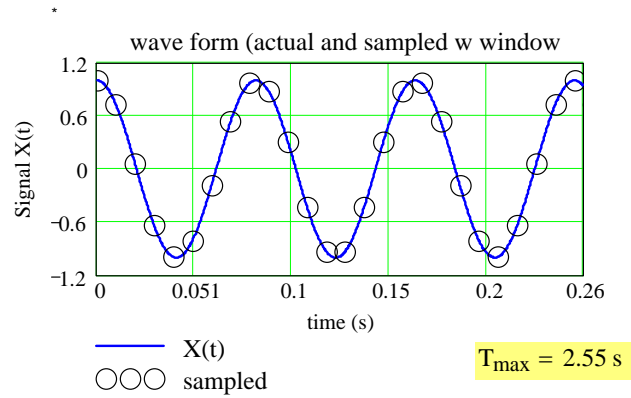


Fig. 1(a): 22 Hz signal sampled at 100 samples/s.

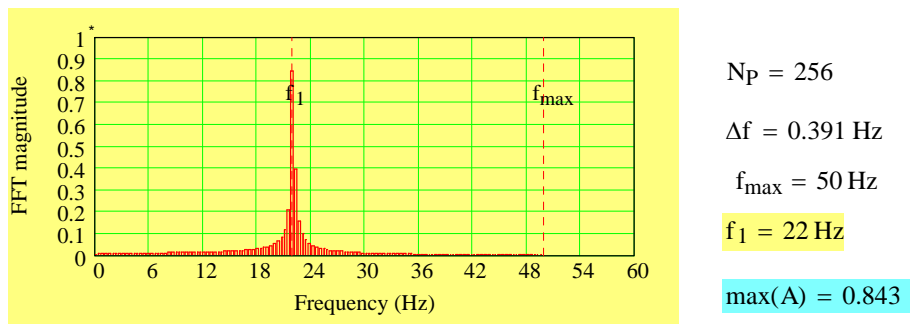


Fig. 1(b): amplitude of DFT for 22 Hz signal sampled at 100 samples/s.

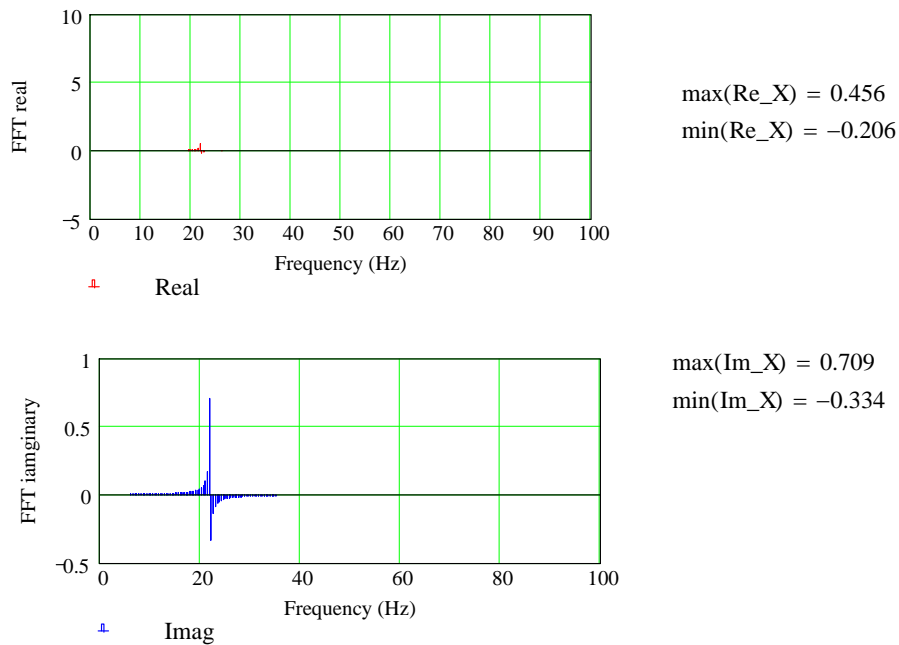


Fig. 1(c): Real and imaginary parts of DFT for 22 Hz signal sampled at 100 samples/s.

The ideal case should be a single amplitude $X=1$ at 22 Hz and 0's at all other frequencies. This ideal representation only occurs when sampling at a frequency that is a multiple of the signal frequency, as shown in Fig 1(d) for sampling at 88 Hz.

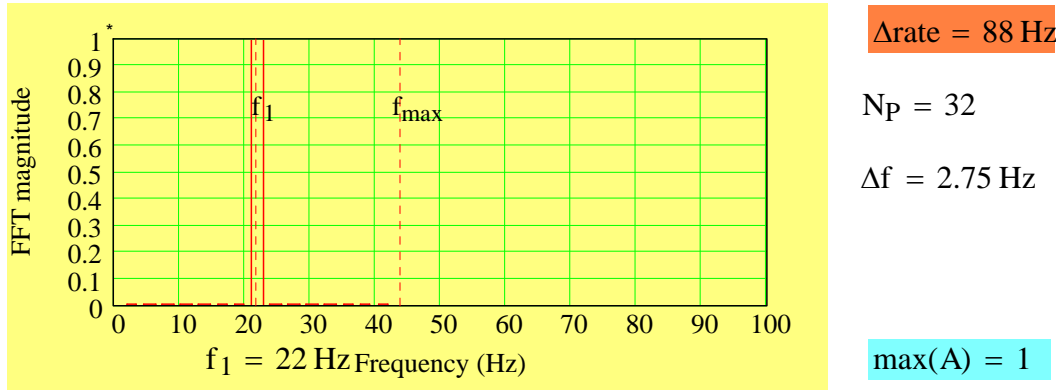


Fig. 1(d): amplitude of DFT for 22 Hz signal sampled at 88 samples/s.

$$f_{\text{req}}^T = \begin{array}{c|cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 1 & 0 & 2.75 & 5.5 & 8.25 & 11 & 13.75 & 16.5 & 19.25 & 22 & 24.75 \end{array} \text{ Hz}$$

$$X^T = \begin{array}{c|cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Notes

- 1) increasing the number of recorded data points N , while keeping the same sampling rate, increases the total time (T) for sampling, but has no impact on the span of the frequency range (f_{max} is the same), only on Δf that decreases (the frequency resolution increases).
- 2) increasing the sampling rate (f_{sample}) while keeping N extends the span of the frequency range ($f_{\text{max}} = \frac{1}{2} f_{\text{sample}}$), and also increases the frequency step Δf (decreases resolution, it makes Δf larger). Increasing f_{sample} , decreases the total elapsed time for measurement.

The table below shows verifies the relationships $f_{\text{max}} = \frac{1}{2} f_{\text{sample}}$ and $(f_{\text{max}} / \Delta f) = k = \frac{1}{2} N$.

| N | f_{sample} (Hz) | f_{max} (Hz) | Δf (Hz) | T (s) |
|-----------|--------------------------|-----------------------|-----------------|---------|
| $2^5=32$ | 40 | 20 | 1.250 | 0.775 |
| $2^6=64$ | 40 | 20 | 0.625 | 1.575 |
| $2^7=128$ | 40 | 20 | 0.313 | 3.175 |
| $2^6=64$ | 40 | 20 | 0.625 | 1.575 |
| $2^6=64$ | 80 | 40 | 1.250 | 0.788 |
| $2^6=64$ | 160 | 80 | 2.500 | 0.394 |

ALIASING

Figure 2(a) shows the same function $x(t) = 1 \sin(\omega t)$, with $\omega = 2\pi f$, $f = 22$ Hz, sampled at 30 Hz (samples/s) or $\Delta t = 0.033$ s, and the number of points is $N = 2^8 = 256$ ($T_{max} = 8.5$ s). Note that $\Delta t \sim 0.045$ s, the period of the 22 Hz wave.

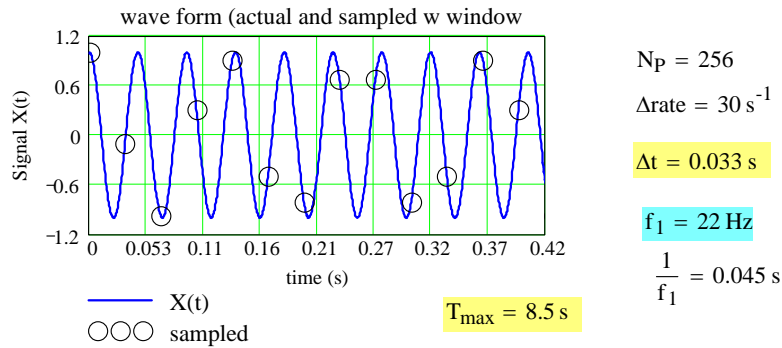


Fig. 2(a): 22 Hz wave sampled at 30 samples/s.

As shown in Fig. 2(b) depicting the amplitude of the DFT, when a 22 Hz sinusoidal signal is sampled at 30 Hz, the sampled data can be misinterpreted as an 8 Hz sinusoidal signal. This is referred to as **aliasing**. Thus, the sampling frequency should be at least **44 samples/s (22 Hz Nyquist)** in order to avoid this problem.

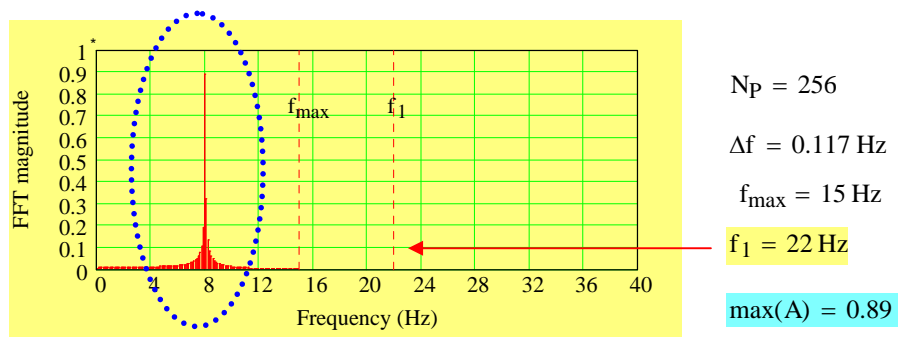


Fig. 2(a): DFT of 22 Hz wave sampled at 30 samples/s.

Leakage

Consider a case where a continuous signal with main frequency 12 Hz is sampled at a frequency of 100 samples/s, and the number of the total sampled data is $N = 32$, as shown in Fig. 3(a). Note in Fig. 3(b) the amplitude of the DFT with components at other frequencies than 12 Hz, including 0 frequency.

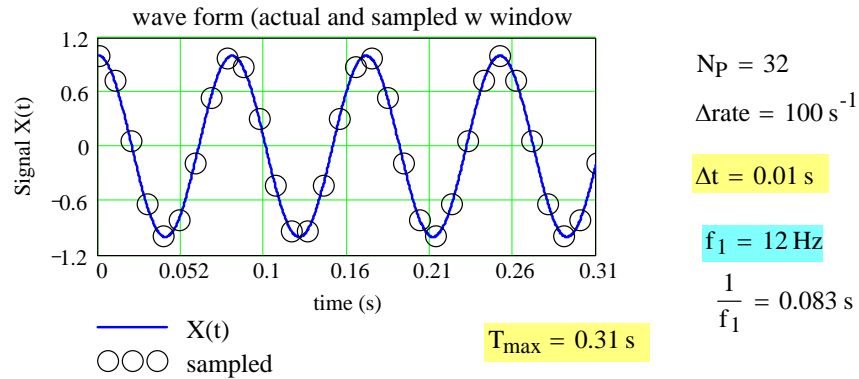


Fig. 3(a): 12 Hz wave sampled at 100 samples/s.

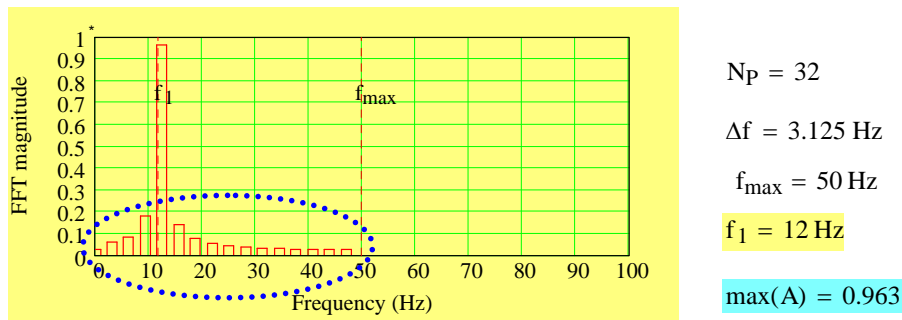


Fig. 3(b): Amplitude of DFT for 12 Hz wave sampled at 100 Hz.

The amplitudes at near zero-frequencies (i.e., the first data points in Fig. 18-3) show **leakage** and is caused by the truncation of time data. That is, the time data at $t = 0$ and $t = T$ have non-zero amplitudes, see Fig. 3(a).

To reduce the truncation error and leakage effect, a **Hanning window**¹ is introduced. The window is defined as

$$H_m = \frac{1}{2} \left[1 - \cos \left(\frac{2\pi m}{N} \right) \right]. \quad (8)$$

and displayed below in Fig. 4.

¹ There are many different types of windows or windowing procedures. Refer to a more advanced resource for details on their implementation and accuracy.

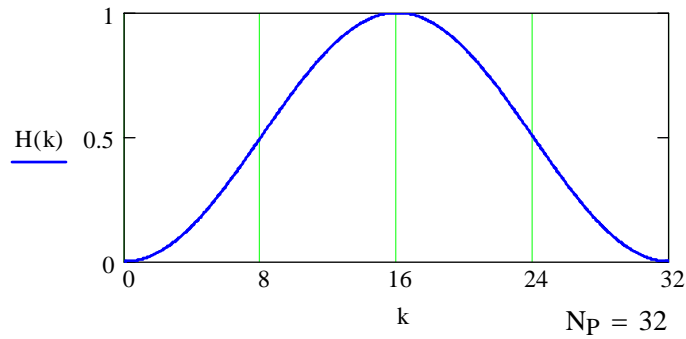


Fig. 4. Hanning window with 32 data points.

Figure 5 shows the signal data set x_n weighted with the Hanning window. The DFT of a windowed time data is

$$X_m = \sum_{n=0}^{N-1} w_n x_n e^{-i2\pi \frac{mn}{N}}, \quad (9)$$

where w_n represents the window function. Based on the window function, two constants are defined as

$$\alpha_1 = \sum_{n=1}^{N-1} w_n \quad \text{and} \quad \alpha_2 = \sum_{n=1}^{N-1} w_n^2 \quad (10)$$

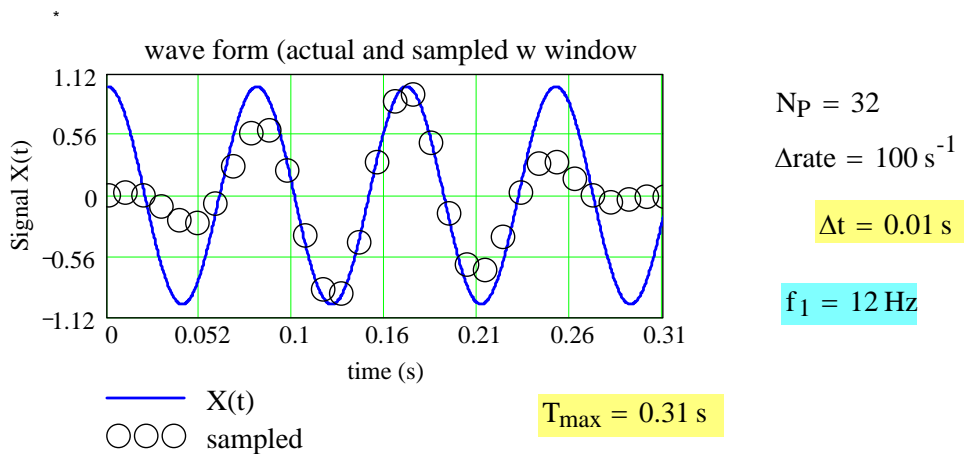
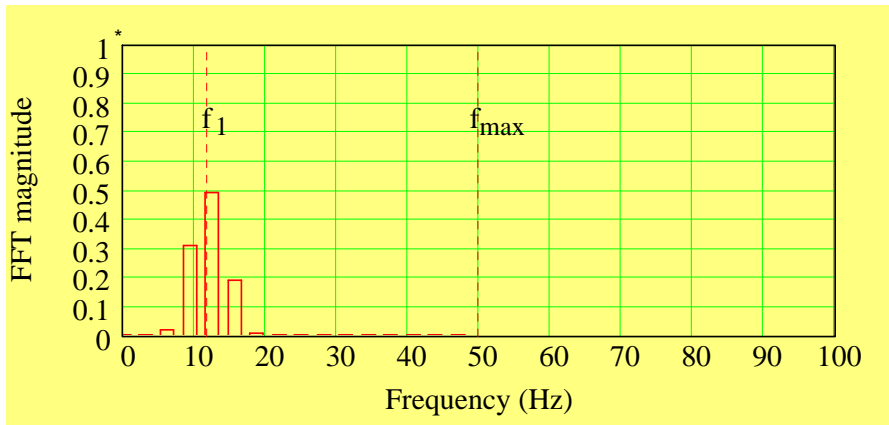


Fig. 5: Sampled 12 Hz wave (100 samples/s) with Hanning window.

At $t = 0$ and $t = T$, the amplitude of the signal = 0. In the frequency domain, as shown in Fig. 6, the leakage of the windowed data is smaller than that for the original data (see Fig. 3(b)), although the frequency resolution of the windowed data is lower than the original data (i.e., the peaks of the windowed data become broader than the original data).



$$N_p = 32$$

$$\Delta f = 3.125 \text{ Hz}$$

$$f_{\max} = 50 \text{ Hz}$$

$$f_1 = 12 \text{ Hz}$$

$$\max(A) = 0.492$$

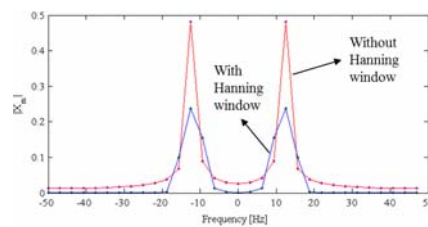


Fig. 6: Amplitude of DFT with Hanning window for 12 Hz wave sampled at 100 Hz.

Spectrum and Spectral Density

All experimental data contains noise. Spectral averaging is applied to reduce the effects of noise.

The **cross-spectrum** of two signals X and Y is (think of a dot product or projection of one signal onto the other)

$$S_{xy_m} = \frac{2X_m^* Y_m}{\alpha_1^2}, \quad m = 0, 1, \dots, k = \frac{N}{2} \quad (11)$$

where X_m^* is the complex conjugate of X_m .

The **auto-spectrum** is also defined as

$$S_{xx_m} = \frac{2X_m^* X_m}{\alpha_1^2}, \quad m = 0, 1, \dots, k = \frac{N}{2} \quad (12)$$

The **cross-spectral density** is defined as

$$\text{CSD}_{xy_m} = \frac{2X_m^* Y_m}{f_{\text{sample}} \alpha_2}, \quad m = 0, 1, \dots, k = \frac{N}{2} \quad (13)$$

The cross spectral density is the spectrum per unit frequency interval.

Spectral Estimation

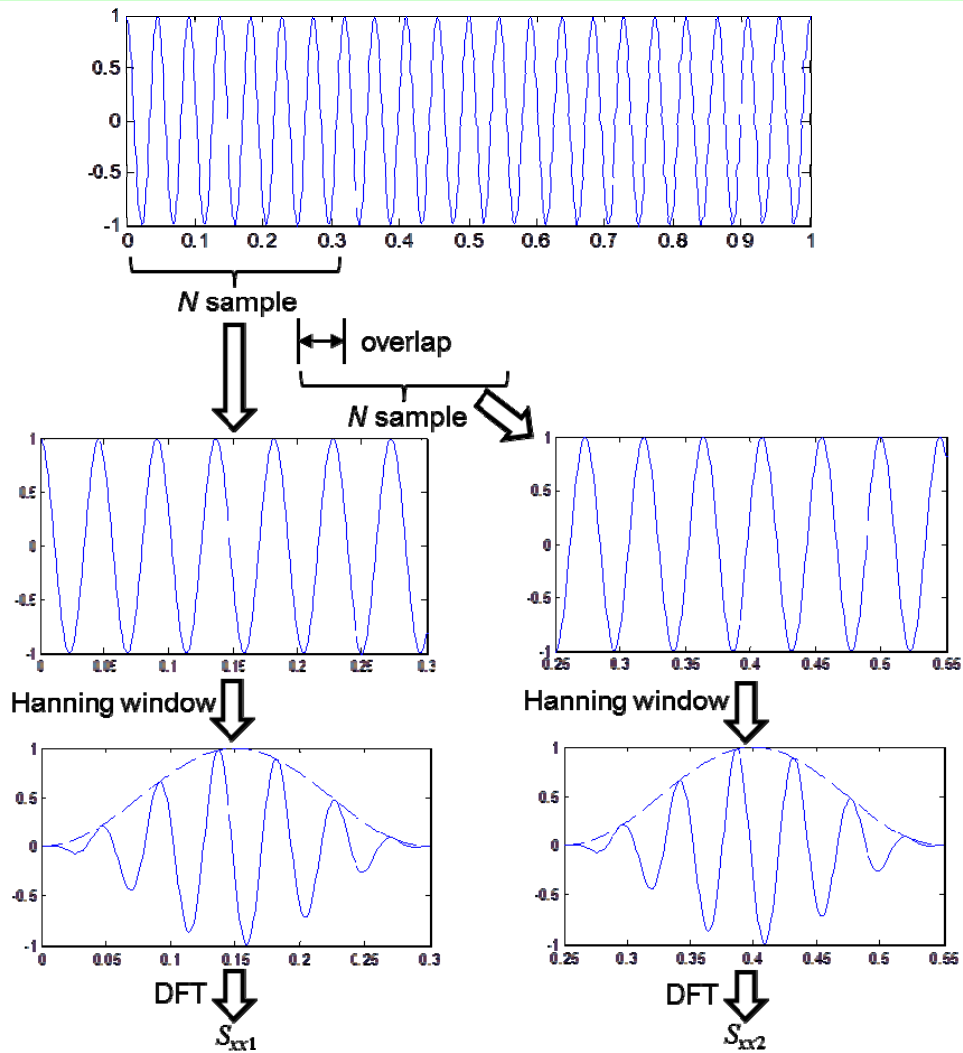


Fig. 7: Averaging process of time data.

Based on the procedure shown in Fig. 7, when the maximum number of averaging is N_a , the spectral averaging process is represented as

$$\bar{S}_{xx} = \frac{1}{N_a} \sum_{m=1}^{N_a} S_{xx(m)}. \quad (14)$$

When the **statistical properties** of a signal do NOT change with respect to time, the signal is referred to as a “**stationary**” signal. Thus, (random) noise effects can be reduced by using a time averaging process, as shown in Fig. 7 and Eq. (14) for any stationary signals.

Transfer Function Estimation

Figure 8 shows a single input and single output (SISO) system with transfer function H .

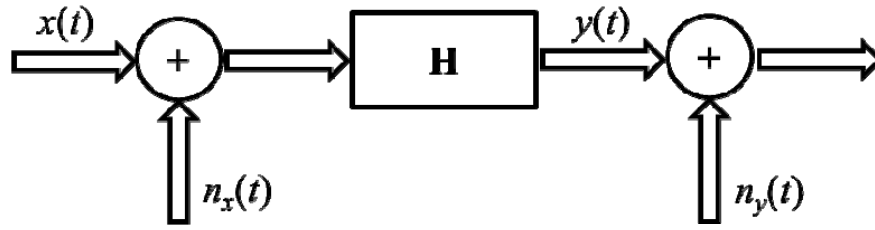


Fig. 8: Depiction of SISO system with transfer function H .
 x : input, y : output, and n : noise

In an **ideal case without measurement noise**, the transfer function is expressed as

$$H = \frac{Y_{(\omega)}}{X_{(\omega)}}. \quad (15)$$

where $X_{(\omega)} = DFT(x_{(t)})$ and $Y_{(\omega)} = DFT(y_{(t)})$. However, when noise components n_x and n_y are present at the input and output of the system, one records the input and output signals as $\bar{x}_{(t)} = x_{(t)} + n_{x(t)}$, $\bar{y}_{(t)} = y_{(t)} + n_{y(t)}$, respectively. Hence, the transfer function becomes

$$H_{(\omega)} = \frac{\bar{Y}_{(\omega)}}{\bar{X}_{(\omega)}} = \frac{Y_{(\omega)} + N_{y(\omega)}}{X_{(\omega)} + N_{x(\omega)}}. \quad (16)$$

Here, the **estimated transfer function is biased due to the noise**.

To estimate an **accurate** transfer function, the noise components must be suppressed. Two types of transfer function estimators are introduced. The first type uses a **cross-spectral correlation with respect to the input**.

$$H_{m1} = \frac{(\bar{X}^* \bar{Y})}{(\bar{X}^* \bar{X})} = \frac{(X + N_x)^* (Y + N_y)}{(X + N_x)^* (X + N_x)} = \frac{S_{xy} + S_{xn_y} + S_{n_x y} + S_{n_x n_y}}{S_{xx} + S_{xn_x} + S_{n_x x} + S_{n_x n_x}}. \quad (17)$$

When the input $x_{(t)}$ and output $y_{(t)}$ are not correlated with either noise (input) n_x and (output) n_y , $(S_{xn_y} = 0, S_{n_x y} = 0, S_{xn_x} = 0, S_{n_x x} = 0)$, and further the noise (n_x, n_y) are not correlated to each other $(S_{n_x n_y} = 0)$, the estimator of the transfer function can be simplified, after taking the time average, as

$$H_{m1} = \frac{S_{xy}}{S_{xx} + S_{n_x n_x}}. \quad (18)$$

Similarly, the second type of estimator uses the **cross-spectral correlation with respect to the**

output

$$H_{m2} = \frac{(\overline{Y^*Y})}{(\overline{Y^*X})} = \frac{(Y + N_y)^*(Y + N_y)}{(Y + N_y)^*(X + N_x)} = \frac{S_{yy} + S_{yn_y} + S_{n_yy} + S_{n_yn_y}}{S_{yx} + S_{yn_x} + S_{n_yx} + S_{n_yn_x}}. \quad (19)$$

It can be simplified to

$$H_{m2} = \frac{S_{yy} + S_{n_yn_y}}{S_{xy}^*}. \quad (20)$$

This **estimator has no bias error** if the noise is present only in the **input signal (x)**; i.e., $S_{n_yn_y} \approx 0$.

Thus, the second type estimator becomes

$$H_{m2} = \frac{S_{yy}}{S_{xy}^*}. \quad (21)$$

This estimator is good at resonance frequencies of a system where the output signal has a large signal to noise ratio (SNR).

Similarly, the first kind estimator has no bias error when the **uncorrelated** noise is present only in the **output signal (y)**, i.e., $S_{n_xn_x} \approx 0$.

Then, the first type estimator becomes

$$H_{m1} = \frac{S_{xy}}{S_{xx}}. \quad (22)$$

This estimator is good at **anti-resonance frequencies** of a system.

Final note:

More on estimations of transfer functions for actual physical systems (experimental data) will follow as the class progresses.