MEEN 617 Handout #11

MODAL ANALYSIS OF MDOF Systems with VISCOUS DAMPING ^ Symmetric

The motion of a *n*-DOF linear system is described by the set of 2nd order differential equations

$$\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{C}\,\dot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{F}_{(t)} \qquad (1)$$

where $\mathbf{U}_{(t)}$ and $\mathbf{F}_{(t)}$ are *n* rows vectors of displacements and external forces, respectively. **M, K, C** are the system (nxn) matrices of mass, stiffness, and viscous damping coefficients. These matrices are symmetric, i.e. $\mathbf{M}=\mathbf{M}^T$, $\mathbf{K}=\mathbf{K}^T$, $\mathbf{C}=\mathbf{C}^T$.

The solution to Eq. (1) is determined uniquely if vectors of initial displacements $\mathbf{U_o}$ and initial velocities $\mathbf{V_o} = \left(d\,\mathbf{U} / d\,t \right)_{t=0}$ are specified.

For **free vibrations**, the force vector $F_{(t)}=0$, and Eq. (1) is

$$\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{C}\,\dot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{0} \qquad (2)$$

A solution to Eq. (2) is of the form

$$\mathbf{U} = e^{\alpha t}\,\boldsymbol{\psi} \qquad (3)$$

where in general α is a complex number. Substitution of Eq. (3) into Eq. (2) leads to the following characteristic equation:

$$\left(\alpha^2\,\mathbf{M} + \alpha\,\mathbf{C} + \mathbf{K} \right)\boldsymbol{\Psi} = \left[\mathbf{f}_{(\alpha)} \right]\boldsymbol{\Psi} = 0 \qquad (4)$$

where $\left[\mathbf{f}_{(\alpha)}\right]$ is a nxn square matrix. The system of homogeneous equations (4) has a nontrivial solution only if the determinant of the system of equation equals zero, i.e.

$$\Delta(\alpha) = \left|\mathbf{f}_{(\alpha)}\right| = 0 = c_0 + c_1\,\alpha + c_2\,\alpha^2 + c_3\,\alpha^3 + \ldots c_{2n}\,\alpha^{2n} \qquad (5)$$

The roots of the characteristic polynomial $\Delta(\alpha)$ given by Eq. (5) can be of three types:

a) **Real and negative,** $\alpha < 0$, corresponding to over damped modes.
b) **Purely imaginary,** $\alpha = \pm i\,\omega$, for undamped modes.
c) **Complex conjugate pairs**[1] **of the form,** $\alpha = \zeta\omega \pm i\,\omega_d$, for under damped modes.

Clearly if the real part of any $\alpha > 0$, it means the system is unstable.

The constituent solution, Eq. (3), $\mathbf{U} = e^{\alpha t}\,\boldsymbol{\psi}$ can be written as the **superposition of the solution roots** e^{α_r} and its associated vectors $\boldsymbol{\Psi}_r$ satisfying Eq. (4), i.e.,

$$\mathbf{U}_{(t)} = \sum_{1}^{2n} C_r\,\boldsymbol{\Psi}_r\,e^{\alpha_r t} \qquad (6)$$

or letting
$$\left[\boldsymbol{\Psi}\right]_{nx2n} = \left[\boldsymbol{\Psi}_1\,\boldsymbol{\Psi}_2 \ldots \boldsymbol{\Psi}_{2n}\right] \qquad (7)$$

write Eq. (6) as

[1] Only if the system is defined by symmetric matrices. Otherwise, the complex roots **may NOT BE** complex conjugate pairs.

$$\mathbf{U}_{(t)} = [\mathbf{\Psi}]\{C_r\, e^{\alpha_r t}\} \tag{8}$$

However, a transformation of the form,

$$\mathbf{U}_{nx1} = [\mathbf{\Psi}]\ \mathbf{q}_{(t)_{2nx1}} \tag{9}$$

is not possible since this implies the existence of **2*n*- modal coordinates** which is not physically apparent when the **number of physical coordinates is only *n***.

To overcome this apparent difficulty, reformulate the problem in a slightly different form. Let **Y** be a 2*n*- rows vector composed of the physical velocities and displacements, i.e.

$$\mathbf{Y} = \begin{bmatrix} \dot{\mathbf{U}} \\ \mathbf{U} \end{bmatrix} \quad \text{, and} \quad \mathbf{Q} = \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_{(t)} \end{bmatrix} \tag{10}$$

be a modified force vector. Then write $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{F}_{(t)}$ as

$$\begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix} \begin{pmatrix} \ddot{\mathbf{U}} \\ \dot{\mathbf{U}} \end{pmatrix} + \begin{bmatrix} \mathbf{-M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{U}} \\ \mathbf{U} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{F} \end{pmatrix} \tag{11.a}$$

or

$$\mathbf{A}\ \dot{\mathbf{Y}} + \mathbf{B}\ \mathbf{Y} = \mathbf{Q} \tag{11.b}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{-M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \tag{12}$$

A and **B** are 2nx2*n* matrices, **symmetric** if the **M, C, K** matrices are also symmetric.

For **free vibrations**, **Q=0**, and a solution to Eq. (11.b) is sought of the form:

$$\begin{bmatrix} \dot{\mathbf{U}} \\ \mathbf{U} \end{bmatrix} = \mathbf{Y} = \mathbf{\Phi}\, e^{\alpha t} \tag{13}$$

Substitution of Eq. (13) into Eq. (11.b) gives:

$$\left[\alpha\,\mathbf{A} + \mathbf{B}\right]\mathbf{\Phi} = 0 \tag{14}$$

which can be written in the familiar form:

$$\mathbf{D}\,\mathbf{\Phi} = \frac{1}{\alpha}\mathbf{\Phi} \tag{15}$$

where

$$\mathbf{D} = \mathbf{-B}^{-1}\mathbf{A} = \begin{pmatrix} \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-K}^{-1} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{pmatrix}, \text{ or } \mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{-K}^{-1}\mathbf{M} & \mathbf{-K}^{-1}\mathbf{C} \end{pmatrix} \tag{16}$$

with **I** as the *n*x*n* identity matrix. From Eq. (15) write

$$\left[\mathbf{D} - \frac{1}{\alpha}\mathbf{I}\right]\mathbf{\Phi} = \left[\mathbf{f}_{(\alpha)}\right]\mathbf{\Phi} = 0 \tag{17}$$

The eigenvalue problem has a nontrivial solution if

$$\Delta(\alpha) = \left|\mathbf{f}_{(\alpha)}\right| = 0 \tag{18}$$

From Eq. (18) determine $2n$ eigenvalues $\{\alpha_r\}$, $r=1, 2.., 2n$ and associated eigenvectors $\{\mathbf{\Phi}_r\}$. Each eigenvector must satisfy the relationship:

$$\mathbf{D}\,\mathbf{\Phi}_r = \frac{1}{\alpha_r}\mathbf{\Phi}_r \qquad (19)$$

and can be written as $\mathbf{\Phi}_\mathbf{r} = \begin{bmatrix} \mathbf{\Psi}_\mathbf{r}^1 \\ \mathbf{\Psi}_\mathbf{r}^2 \end{bmatrix}$ where $\mathbf{\Psi}_\mathbf{r}$ is a nx1 vector satisfying:

$$\begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{-K^{-1}M} & \mathbf{-K^{-1}\,C} \end{pmatrix} \begin{bmatrix} \mathbf{\Psi}_\mathbf{r}^1 \\ \mathbf{\Psi}_\mathbf{r}^2 \end{bmatrix} = \frac{1}{\alpha_r} \begin{bmatrix} \mathbf{\Psi}_\mathbf{r}^1 \\ \mathbf{\Psi}_\mathbf{r}^2 \end{bmatrix} \qquad (20)$$

from the first row of Eq. (20) determine that:

$$\mathbf{I}\;\mathbf{\Psi}_\mathbf{r}^2 = \frac{1}{\alpha_r}\,\mathbf{\Psi}_\mathbf{r}^1 \quad \text{or} \quad \mathbf{\Psi}_\mathbf{r}^1 = \alpha_r\;\mathbf{\Psi}_\mathbf{r}^2 \qquad (21)$$

and from the second row of Eq. (20), with substitution of the relationship in Eq. (21), obtain $\mathbf{-K^{-1}M}\,\mathbf{\Psi}_\mathbf{r}^1 - \mathbf{K^{-1}\,C}\,\mathbf{\Psi}_\mathbf{r}^2 = \frac{1}{\alpha_r}\mathbf{\Psi}_\mathbf{r}^2$ or

$$\left[\left(\mathbf{-K^{-1}M}\right)\alpha_r - \left(\mathbf{K^{-1}C}\right) - \mathbf{I}\frac{1}{\alpha_r} \right] \mathbf{\Psi}_\mathbf{r}^2 = \mathbf{0} \qquad (23)$$

for $r=1, 2,\ldots 2n$. Note that multiplying Eq. (23) by $(-\alpha_r\,\mathbf{K})$ gives

$$\left[\mathbf{M}\,\alpha_r^2 + \mathbf{C}\,\alpha_r + \mathbf{K} \right] \mathbf{\Psi}_\mathbf{r}^2 = \mathbf{0} \qquad (4)$$

i.e., the **original eigenvalue problem**.

Solution of Eq. (19), $\mathbf{D}\boldsymbol{\Phi}_r = \frac{1}{\alpha_r}\boldsymbol{\Phi}_r$, delivers the ***2n-eigenpairs***

$$\left(\alpha_r \,;\, \boldsymbol{\Phi}_\mathbf{r} = \begin{bmatrix} \alpha_r \boldsymbol{\Psi}_\mathbf{r} \\ \boldsymbol{\Psi}_\mathbf{r} \end{bmatrix}\right)_{r=1,2,..2n} \tag{24}$$

In general, the j-components of the eigenvectors $\boldsymbol{\Psi}_\mathbf{r}$ are complex numbers written as

$$\boldsymbol{\Psi}_{\mathbf{r}_j} = a_{rj} + i\, b_{rj} = \delta_{rj}\, e^{i\phi_{rj}} \quad \text{j=1,2}\ldots n$$

where δ and φ denote the magnitude and the phase angle.

Note: for viscous damped systems, not only the amplitudes but also the phase angles are arbitrary. However, the ratios of amplitudes and differences in phase angles are constant for each of the elements in the eigenvector $\boldsymbol{\Psi}_\mathbf{r}$. That is,

$$\left(\delta_j / \delta_k\right) = \text{const}_{jk} \text{ and} \left(\phi_j - \phi_k\right) = \text{const}_{jk} \text{ for } j,\, k = 1,\, 2,\, ..N$$

A constituent solution of the homogeneous equation (**free vibration** problem) is then given as:

$$\mathbf{Y} = \begin{bmatrix} \dot{\mathbf{U}} \\ \mathbf{U} \end{bmatrix} = \sum_{r=1}^{2n} C_r\, e^{\alpha_r t}\, \boldsymbol{\Phi}_\mathbf{r} \tag{25}$$

Let the (roots) α_r be written in the form (when under-damped)

$$\alpha_r = \zeta_r\, \omega_r + i\, \omega_{d\,r} \tag{26}$$

and write Eq. (25) as

$$\mathbf{Y} = \begin{bmatrix} \dot{\mathbf{U}} \\ \mathbf{U} \end{bmatrix} = \sum_{r=1}^{2n} C_r \, \mathbf{\Phi_r} \; e^{(-\zeta_r \omega_r + i\omega_{dr})\,t} \qquad (27)$$

and since $\mathbf{\Phi_r} = \begin{bmatrix} \alpha_r \mathbf{\Psi_r} \\ \mathbf{\Psi_r} \end{bmatrix}$, the vector of displacements is just

$$\mathbf{U} = \sum_{r=1}^{2n} C_r \, \mathbf{\Psi_r} \; e^{(-\zeta_r \omega_r + i\omega_{dr})\,t} \qquad (27)$$

ORTHOGONALITY OF DAMPED MODES

Each eigenvalue α_r and its corresponding eigenvector $\mathbf{\Phi_r}$ satisfy the equation:

$$\alpha_r \; \mathbf{A} \; \mathbf{\Phi_r} + \mathbf{B} \; \mathbf{\Phi_r} = 0 \qquad (28)$$

Consider two different eigenvalues (not complex conjugates): $\{\alpha_s ; \mathbf{\Phi_s}\}$ and $\{\alpha_q ; \mathbf{\Phi_q}\}$, then **if** $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$ and $\mathbf{B} = \mathbf{B}^{T}$ (a symmetric system), it is easy to demonstrate that:

$$\left(\alpha_s - \alpha_q\right) \mathbf{\Phi_s^T} \; \mathbf{A} \, \mathbf{\Phi_q} = 0$$

and infer

$$\mathbf{\Phi_s^T} \, \mathbf{A} \, \mathbf{\Phi_q} = \mathbf{0} \; ; \; \mathbf{\Phi_s^T} \, \mathbf{B} \, \mathbf{\Phi_q} = \mathbf{0} \quad \text{for } \alpha_s \neq \alpha_q \qquad (29)$$

Now, construct a **modal damped matrix** $\mathbf{\Phi}$ $_{(2n\text{x}2n)}$ formed by the columns of the modal vectors $\mathbf{\Phi_r}$, i.e.

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 & \ldots & \mathbf{\Phi}_n & .. & \mathbf{\Phi}_{2n-1} & \mathbf{\Phi}_{2n} \end{bmatrix} \qquad (30)$$

And write the **orthogonality property** as:

$$\mathbf{\Phi}^T \mathbf{A}\,\mathbf{\Phi} = \boldsymbol{\sigma} \qquad \mathbf{\Phi}^T \mathbf{B}\,\mathbf{\Phi} = \boldsymbol{\beta} \tag{31}$$

Where $\boldsymbol{\sigma}$ and $\boldsymbol{\beta}$ are $(2n\text{x}2n)$ diagonal matrices.

Now, recall that the equations of motion in physical coordinates are :

$$\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{C}\,\dot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{F}_{(t)} \tag{1}$$

With the definition $\mathbf{Y} = \begin{bmatrix} \dot{\mathbf{U}} \\ \mathbf{U} \end{bmatrix}$, Eqs. (1) are converted into $2n$ first order differential equations:

$$\mathbf{A}\,\dot{\mathbf{Y}} + \mathbf{B}\,\mathbf{Y} = \mathbf{Q} = \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_{(t)} \end{bmatrix} \tag{32}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \tag{12}$$

To uncouple the set of $2n$ first-order Eqs. (32), a solution of the following form is assumed:

$$\mathbf{Y}_{(t)} = \sum_{r=1}^{2n} \mathbf{\Phi}_{\mathbf{r}}\, z_{r(t)} = \mathbf{\Phi}\,\mathbf{Z}_{(t)} \tag{33}$$

Substitution of Eq. (33) into Eq. (32) gives:

$$\mathbf{A}\ \mathbf{\Phi}\ \dot{\mathbf{Z}} + \mathbf{B}\ \mathbf{\Phi}\ \mathbf{Z} = \mathbf{Q} \tag{34}$$

Premultiply this equation by $\mathbf{\Phi}^{\mathbf{T}}$ and use the orthogonality property[2] of the damped modes to get:

$$\left(\mathbf{\Phi}^{\mathbf{T}}\mathbf{A}\ \mathbf{\Phi}\right)\dot{\mathbf{Z}}+\left(\mathbf{\Phi}^{\mathbf{T}}\mathbf{B}\ \mathbf{\Phi}\right)\mathbf{Z}=\mathbf{\Phi}^{\mathbf{T}}\mathbf{Q} \tag{35}$$

or

$$\boldsymbol{\sigma}\ \dot{\mathbf{Z}}+\boldsymbol{\beta}\ \mathbf{Z}=\mathbf{G}=\mathbf{\Phi}^{\mathbf{T}}\ \mathbf{Q} \tag{36}$$

Eq. (36) represents a set of 2*n* uncoupled first order equations:

$$\sigma_1\ \dot{z}_1+\beta_1\ z_1=g_{1_{(t)}}$$

$$\sigma_2\ \dot{z}_2+\beta_2\ z_2=g_{2_{(t)}}$$

......

$$\sigma_{2N}\ \dot{z}_{2N}+\beta_{2N}\ z_{2N}=g_{2N_{(t)}} \tag{37}$$

where $\sigma_r=\mathbf{\Phi}_{\mathbf{r}}^{\mathbf{T}}\ \mathbf{A}\,\mathbf{\Phi}_{\mathbf{r}}\ ;\ \beta_r=\mathbf{\Phi}_{\mathbf{r}}^{\mathbf{T}}\ \mathbf{B}\,\mathbf{\Phi}_{\mathbf{r}}=-\alpha_r\ \sigma_r$, r=1, 2..2N

$$\alpha_r=-\beta_r/\sigma_r \tag{38}$$

since $\alpha_r\ \mathbf{A}\,\mathbf{\Phi}_{\mathbf{r}}+\mathbf{B}\ \mathbf{\Phi}_{\mathbf{r}}=\mathbf{0}$. In addition,

$$g_{r_{(t)}}=\mathbf{\Phi}_{\mathbf{r}}^{\mathbf{T}}\ \mathbf{Q}_{(t)} \tag{39}$$

Initial conditions are also determined from $\mathbf{Y}_o=\begin{bmatrix}\dot{\mathbf{U}}_o\\ \mathbf{U}_o\end{bmatrix}$ with the transformation $\mathbf{Y}_{(0)}=\mathbf{\Phi}\,\mathbf{Z}_{(t=0)}$

$$\boldsymbol{\sigma}\,\mathbf{Z}_{\mathbf{0}}=\mathbf{\Phi}^{\mathbf{T}}\ \mathbf{A}\,\mathbf{Y}_{\mathbf{0}} \tag{40.a}$$

[2] The result below is only valid for symmetric systems, i.e. with **M, K** and **C** as symmetric matrices. For the more general case (non symmetric system), see the textbook of **Meirovitch** to find a discussion on LEFT and RIGHT eigenvectors.

or $$Z_{o_r} = \frac{1}{\sigma_r} \mathbf{\Phi}_\mathbf{r}^\mathbf{T} \mathbf{A} \mathbf{Y_o} \qquad r=1, 2, \ldots\ 2n \qquad (40.b)$$

The general solution of the first order equation $\sigma_r \dot{z}_r + \beta_r z_r = g_r(t)$, with initial condition $z_{r(t=0)} = z_{o_r}$, is derived from the Convolution integral

$$z_r = z_{o_r}\ e^{\alpha_r t} \frac{1}{\sigma_r} \int_0^t g_{r(\tau)}\ e^{\alpha_r (t-\tau)} d\tau \qquad (41)$$

with $\alpha_r = -\beta_r / \sigma_r$

Once each of the $z_{r(t)}$ solutions is obtained, then return to the physical coordinates to obtain:

$$\mathbf{Y}_{(t)} = \begin{bmatrix} \dot{\mathbf{U}} \\ \mathbf{U} \end{bmatrix} = \sum_{r=1}^{2n} \mathbf{\Phi_r}\ z_{r(t)} = \mathbf{\Phi}\, \mathbf{Z}_{(t)} \qquad (33{=}43)$$

and since $\mathbf{\Phi_r} = \begin{bmatrix} \alpha_r \mathbf{\Psi_r} \\ \mathbf{\Psi_r} \end{bmatrix}$, the physical displacement dynamic response is given by:

$$\mathbf{U}_{(t)} = \sum_{r=1}^{2n} \mathbf{\Psi_r}\ z_{r(t)} \qquad (44)$$

and the velocity vector is correspondingly equal to:

$$\dot{\mathbf{U}}_{(t)} = \sum_{r=1}^{2n} \alpha_r\ \mathbf{\Psi_r}\ z_{r(t)} \qquad (45)$$

Read/study the accompanying MATHCAD® worksheet with a detailed example for discussion in class.

MODAL ANALYSIS of MDOF linear systems with viscous damping

Original by Dr. Luis San Andres for MEEN 617 class / SP 08, 12

The equations of motion are: $$M\, d^2U/dt^2 + C\, dU/dt + K\, U = F(t) \quad (1)$$

where M,C,K are nxn SYMMETRIC matrices of inertia, viscous damping and stiffness coefficients, and U, dU/dt, d^2U/dt^2,and are the nx1 vectors of displacements, velocity and accelerations. F(t) is the nx1 vector of generalized forces. Eq (1) is solved with appropriate initial conditions, at t=0, Uo,Vo=dU/dt

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Define elements of inertia, stiffness, and damping matrices:

$m_1 := 100$ $\quad k_1 := 1.0\cdot 10^7 \quad c_1 := 5000$

$m_2 := 100$ kg $\quad k_2 := 1.0\cdot 10^7$ N/m $\quad c_2 := 2000$ N.s/m

$m_3 := 50$ $\quad k_3 := 2.0\cdot 10^7 \quad c_3 := 1000$

$n := 3$ # of DOF

Make matrices:

$$M := \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \quad K := \begin{pmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{pmatrix} \quad C := \begin{pmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{pmatrix}$$

Initial conditions in displacement and velocity:

$$Uo := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad Vo := \begin{pmatrix} 0 \\ .1 \\ 0 \end{pmatrix}$$

PLOTs - only $N := 1024$ # for time steps

natural freqs - undamped

damped eigenvals

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2. Evaluate the damped eigenvalues: Rewrite Eq (1), as

A dY/dt + B Y = Q(t),

where $Y = [\, dU/dt,\, U\,]^T$, and $Q=[0, F_{(t)}]^T$ are 2n row vectors of (velocity, displacements) and generalized forces; and initial conditions Yo = [Vo, Uo]

$$A = \begin{pmatrix} 0 & M \\ M & C \end{pmatrix} \quad B = \begin{pmatrix} -M & 0 \\ 0 & K \end{pmatrix}$$

are 2nx2n Symmetric matrices

2.1 <u>define the A & B augmented matrices:</u>

$\text{zero} := \text{identity}(n) - \text{identity}(n)$ the (nxn) null matrix

$A := \text{stack}(\text{augment}(\text{zero}, M), \text{augment}(M, C))$

$B := \text{stack}(\text{augment}(-M, \text{zero}), \text{augment}(\text{zero}, K))$

2.2 Use MATHCAD function to <u>calculate eigenvectors and eigenvalues of the generalized eigenvalue problem, M X = α N X.</u>

In vibrations problems we set the problem as: $\alpha A \phi + B \phi = 0$,
hence M=-B, N=A to use properly the MATHCAD functions genvecs & genvals

$\alpha := \text{genvals}(B, -A)_{\text{rad/s}}$

$\Phi_D := \text{genvecs}(B, -A)$

$$\alpha = \begin{pmatrix} -5.33 + 165.43i \\ -5.33 - 165.43i \\ -33.31 + 474.21i \\ -33.31 - 474.21i \\ -21.36 + 803.53i \\ -21.36 - 803.53i \end{pmatrix}$$

Recall the undamped natural frequencies

$$\omega = \begin{pmatrix} 165.46 \\ 475.33 \\ 804.17 \end{pmatrix}$$

note that the (damped) eigenvalues are complex conjugates, with the same real part and +/- imaginary parts

$$\Phi_D = \begin{pmatrix} -0.35 - 0.12i & -0.35 + 0.12i & -0.38 + 0.75i \\ -0.6 - 0.21i & -0.6 + 0.21i & 0.07 - 0.21i \\ -0.64 - 0.23i & -0.64 + 0.23i & 0.22 - 0.44i \\ -6.34 \times 10^{-4} + 2.13i \times 10^{-3} & -6.34 \times 10^{-4} - 2.13i \times 10^{-3} & 1.62 \times 10^{-3} + 6.97i \times 10^{-4} \\ -1.18 \times 10^{-3} + 3.64i \times 10^{-3} & -1.18 \times 10^{-3} - 3.64i \times 10^{-3} & -4.48 \times 10^{-4} - 1.14i \times 10^{-} \\ -1.28 \times 10^{-3} + 3.91i \times 10^{-3} & -1.28 \times 10^{-3} - 3.91i \times 10^{-3} & -9.63 \times 10^{-4} - 4.05i \times 10^{-} \end{pmatrix}$$

Note that the eigenvectors are conjugate pairs, i.e. they show the same real part and +/- imaginary part. In addition the first n-rows of an eigenvector are proportional to the 2nd n-rows. The proportionality constant is the damped eingenvalue.

2.2. Form "damped" modal matrices using the orthogonality properties:

$$\sigma := \Phi_D{}^T \cdot A \cdot \Phi_D \qquad \beta := \Phi_D{}^T \cdot B \cdot \Phi_D$$

$$\sigma = \begin{pmatrix} 0.54-0.75i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.54+0.75i & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.26+0.26i & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.26-0.26i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.12+0.11i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.12-0.11i \end{pmatrix}$$

$$\beta = \begin{pmatrix} -121.82-93.33i & 6.53\times10^{-15}-1.82i\times10^{-15} & 9.12i\times10^{-15} \\ 6.53\times10^{-15}+1.82i\times10^{-15} & -121.82+93.33i & 2.54\times10^{-15}+1.29i\times10 \\ 3.17\times10^{-15}+4.86i\times10^{-15} & 2.57\times10^{-15}+1.99i\times10^{-14} & 113.59+132.39i \\ 2.57\times10^{-15}-1.99i\times10^{-14} & 3.17\times10^{-15}-4.86i\times10^{-15} & -1.3\times10^{-14} \\ 4.71\times10^{-15}-2.16i\times10^{-14} & 8.43\times10^{-15}+2.27i\times10^{-14} & -2.45\times10^{-14}+1.33i\times1 \\ 8.43\times10^{-15}-2.27i\times10^{-14} & 4.71\times10^{-15}+2.16i\times10^{-14} & 2.34\times10^{-15}+1.56i\times10 \end{pmatrix}$$

Note off-diagonal terms are small - but NOT zero (as they should)

Check the orthogonality property:

compare β/σ ratios to eigenvalues

$$\frac{\beta_{j,j}}{\sigma_{j,j}} = \begin{pmatrix} 5.33-165.43i \\ 5.33+165.43i \\ 33.31-474.21i \\ 33.31+474.21i \\ 21.36-803.53i \\ 21.36+803.53i \end{pmatrix} \qquad \alpha = \begin{pmatrix} -5.33+165.43i \\ -5.33-165.43i \\ -33.31+474.21i \\ -33.31-474.21i \\ -21.36+803.53i \\ -21.36-803.53i \end{pmatrix}$$

undamped:

$$\omega = \begin{pmatrix} 165.46 \\ 475.33 \\ 804.17 \end{pmatrix}$$

for underdamped systems only

real part: $-\xi\cdot\omega_n$

imaginary part: $\omega_d = \omega_n\cdot\left(1-\xi^2\right)^{.5}$

damping ratios:

$j := 1 .. 2\cdot n$

$$\xi_j := \left[\frac{1}{\left(\frac{Im(\alpha_j)}{Re(\alpha_j)}\right)^2 + 1}\right]^{.5}$$

$$\omega_{n_j} := \frac{Re(\alpha_j)}{-\xi_j}$$

natural frequencies

$$\omega_{d_j} := Im(\alpha_j)$$

damped natural freqs.

damped eigenvals

$\xi^T = (0.03 \quad 0.03 \quad 0.07 \quad 0.07 \quad 0.03 \quad 0.03)$ **damping ratios.**

$\omega_n^T = (165.52 \quad 165.52 \quad 475.37 \quad 475.37 \quad 803.81 \quad 803.81)$ **freqs.** **rad/s**

$\omega_d^T = (165.43 \quad -165.43 \quad 474.21 \quad -474.21 \quad 803.53 \quad -803.53)$ **damped natural freqs.**

$\omega^T = (165.46 \quad 475.33 \quad 804.17)$ ...... undamped modal analysis

damped modal response

3. The 2n first order equations $A\ dY/dt + B\ Y = Q(t)$ with the transformation $Y=\Phi_d Z$ become 2n equations of the form:

$$\sigma_i\ dZ_i/dt + \beta_i\ Z_i = G_i$$ **, i=1,2,3,....2n where**

$G=\Phi_d^T Q(t)$ **and initial conditions** $Zo=\sigma^{-1}\Phi_d^T\ A\ Yo$**:**

3.1 solve the 2n-first order differential equations for the Free Response to initial conditions, F(t)=0:

From initial conditions in displacement and velocity, set $Yo := \text{stack}(Vo, Uo)$

and **in damped modal space:** $Zo := \sigma^{-1}\cdot\left(\Phi_D{}^T\cdot A\cdot Yo\right)$

$p := 1 .. N$ **number of data points**

$m := 2\cdot n$

$t_p := (p-1)\cdot\Delta t$ **time sequence**

get the modal response: $Z_{j,p} := Zo_j\cdot e^{\alpha_j\cdot t_p}$

andback into the physical coordinates: $Y=\Phi_d Z$

$s := 1 .. 2\cdot n$

FREE RESPONSE using DAMPED MODES

$$Y_{s,p} := \sum_{q=1}^{m} \Phi_{D_{s,q}}\cdot Z_{q,p}$$

$j := 1 .. 2\cdot n$

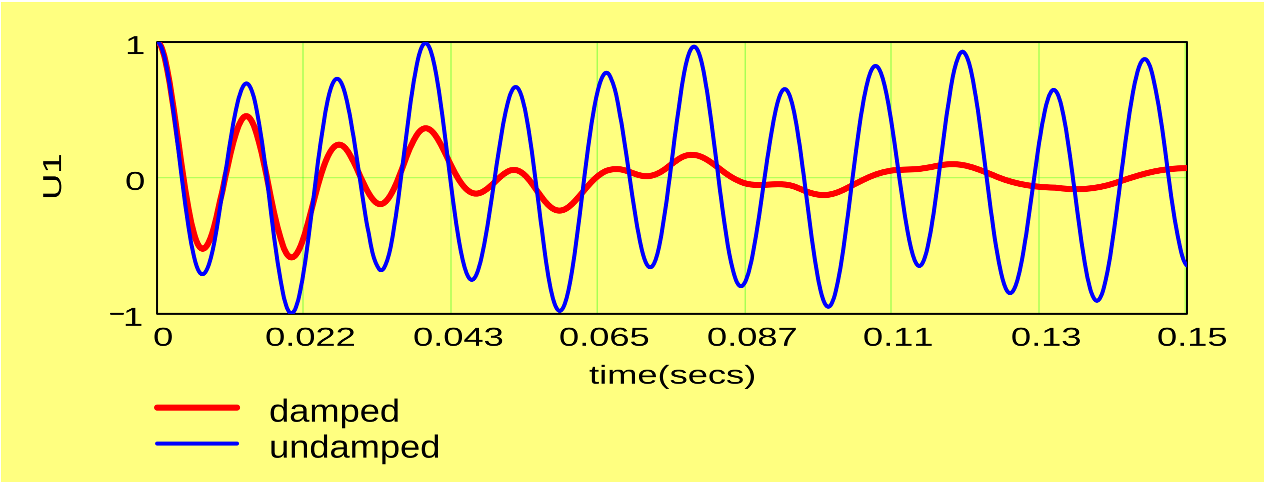
damped modal response

Plot the displacements (last n rows of Y vector):

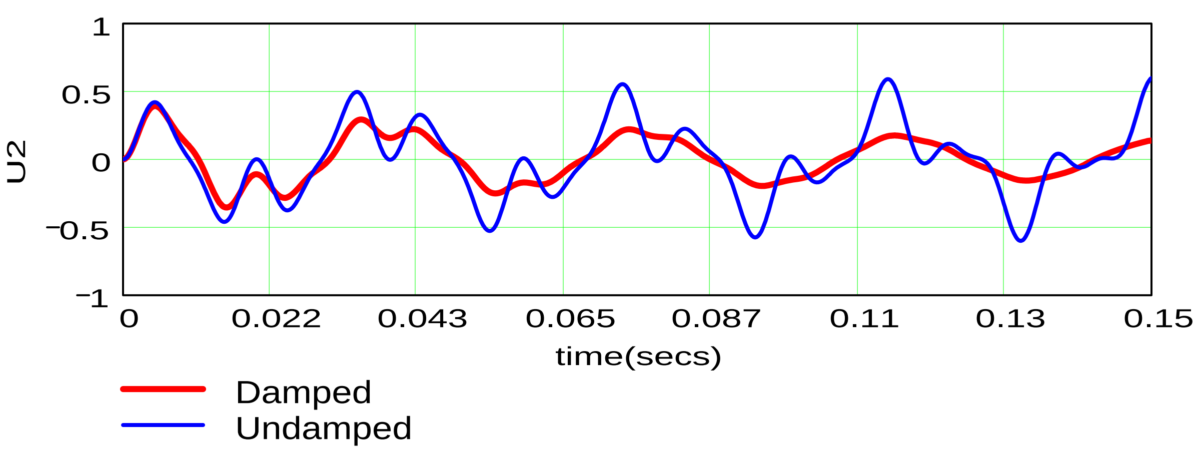
recall:

$$Uo = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

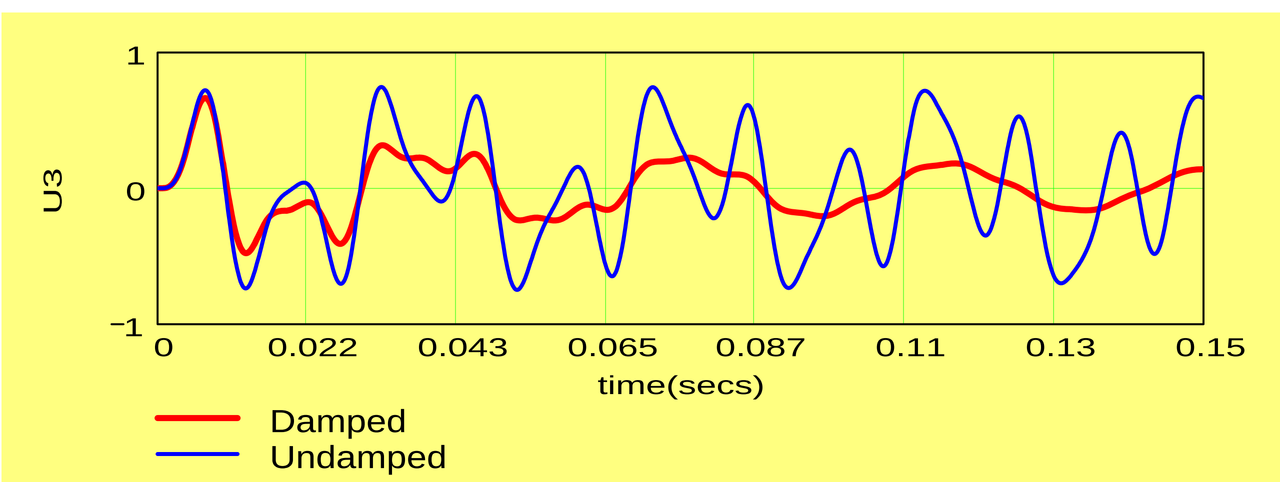
RESPONSE: U1



RESPONSE: U2



RESPONSE: U3



$$\xi^T = (0.03 \quad 0.03 \quad 0.07 \quad 0.07 \quad 0.03 \quad 0.03)$$

4. Forced response to step load F(t)=Fo and initial conditions

Set initial conditions in displacement and velocity:

STEP FORCE

$$Uo := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad Vo := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad Fo := \begin{pmatrix} 30000 \\ 10^5 \\ -10^5 \end{pmatrix} \qquad O := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Step load response

Then set: $Yo := \mathrm{stack}(Vo, Uo)$ $\qquad Qo := \mathrm{stack}(O, Fo)$

and transform to damped modal space: $Zo := \sigma^{-1}\cdot\left(\Phi_D{}^T\cdot A\cdot Yo\right)$ $\qquad Go := \Phi_D{}^T\cdot Qo$

$Tmax := \dfrac{8}{f_1}$ secs $\qquad \Delta t := \dfrac{Tmax}{N} \qquad j := 1 .. 2\cdot n$

$p := 1 .. N$

$t_p := (p-1)\cdot\Delta t$

to get the modal response:

$$Z_{j,p} := Zo_j\cdot e^{\alpha_j\cdot t_p} + \frac{Go_j}{\beta_{j,j}}\cdot\left(1 - e^{\alpha_j\cdot t_p}\right)$$

and in physical coordinates:

$s := 1 .. m \qquad m := 2\cdot n$

STEP RESPONSE
using DAMPED MODES

$Y = \Phi_d Z$

$$Y_{s,p} := \sum_{q=1}^{m} \Phi_{D_{s,q}}\cdot Z_{q,p}$$

===

6. For completeness, obtain also the undamped forced response:

$\eta o := Mm^{-1}\cdot\Phi^T\cdot M\cdot Uo \quad \upsilon o := Mm^{-1}\cdot\Phi^T\cdot M\cdot V P := \Phi^T\cdot Fo$

$j := 1 .. n$

$$\eta_{j,p} := \eta o_j\cdot\cos\left(\omega_j\cdot t_p\right) + \frac{\upsilon o_j}{\omega_j}\cdot\sin\left(\omega_j\cdot t_p\right) + \frac{P_j}{Km_{j,j}}\cdot\left(1 - \cos\left(\omega_j\cdot t_p\right)\right)$$

and into physical coordinates: $\qquad s := 1 .. n$

STEP RESPONSE
using UNDAMPED MODES

$U \equiv \Phi\cdot\eta$

$$U_{s,p} := \sum_{q=1}^{n} \Phi_{s,q}\cdot\eta_{q,p}$$

===

static response - check

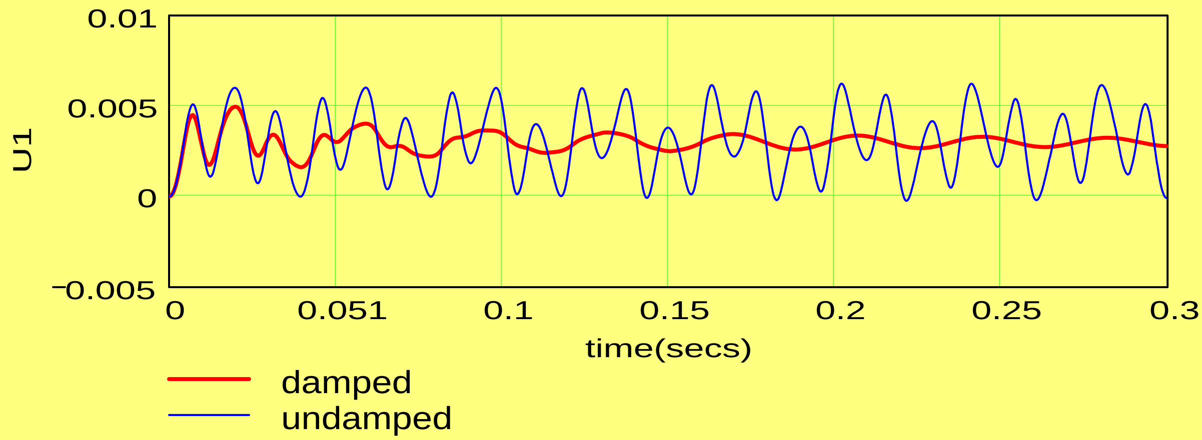
$Us := K^{-1}\cdot Fo$

Plot the displacements (last n rows of Y vector):

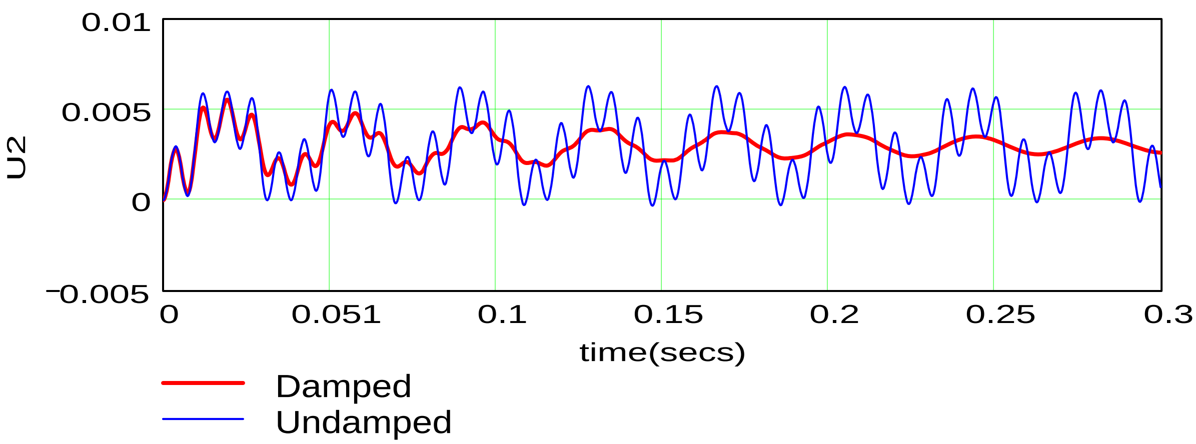
Step load response

RESPONSE: U1

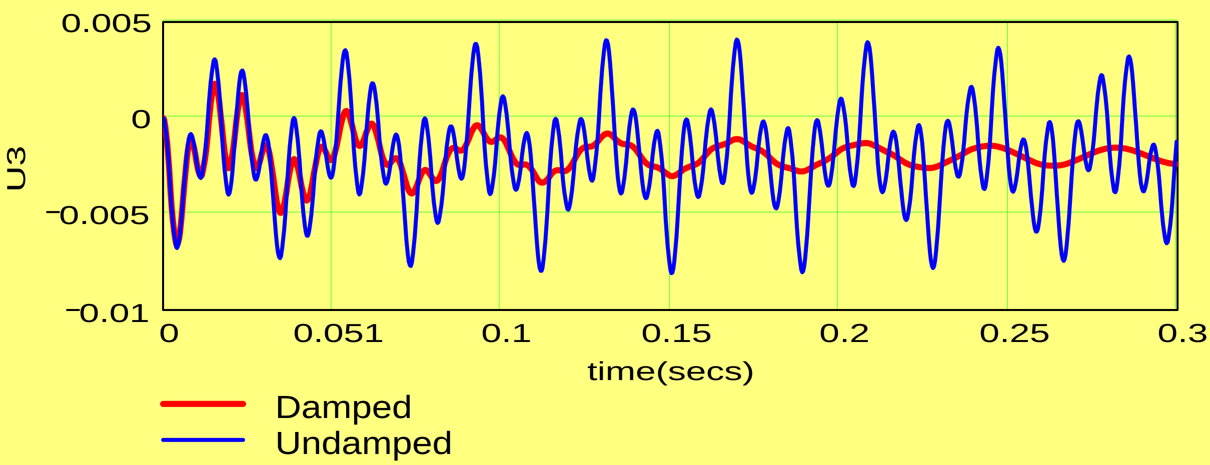
$U_s^T = \left(3 \times 10^{-3} \quad 3 \times 10^{-3} \quad -2 \times 10^{-}\right.$



RESPONSE: U2



RESPONSE: U3



$\xi^T = (0.03 \quad 0.03 \quad 0.07 \quad 0.07 \quad 0.03 \quad 0.03)$

5. Periodic response to loading, F(t)=Fo sin(Ωt):

Response due to initial conditions vanishes after a long time because of damping:

$$Fo = \begin{pmatrix} 3 \times 10^{4} \\ 1 \times 10^{5} \\ -1 \times 10^{5} \end{pmatrix}$$

freq. of excitation

$fHz := 80$

$\Omega := 2 \cdot \pi \cdot fHz$

$\Omega = 502.65$

$$\omega = \begin{pmatrix} 165.46 \\ 475.33 \\ 804.17 \end{pmatrix}$$

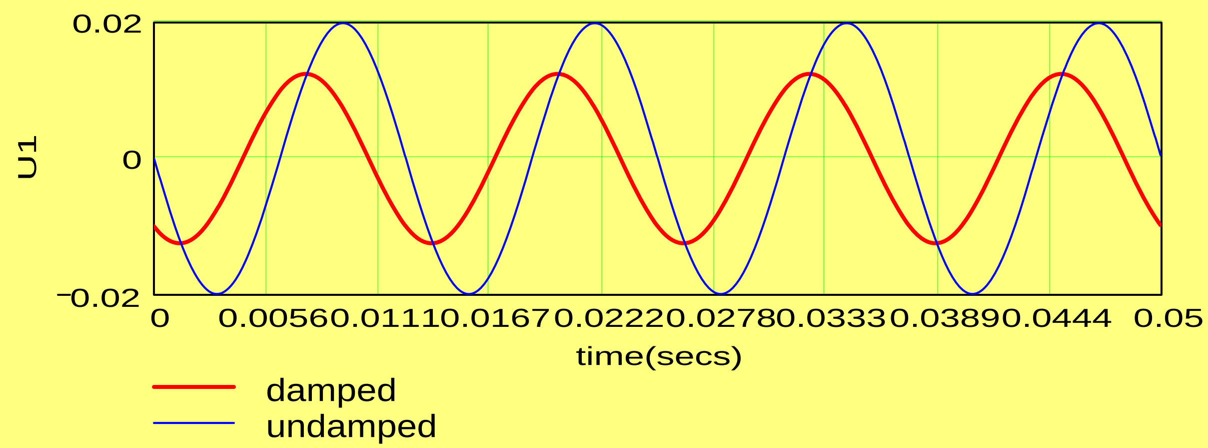
recall the undamped frequencies

Plot the displacements (last n rows of Y vector):

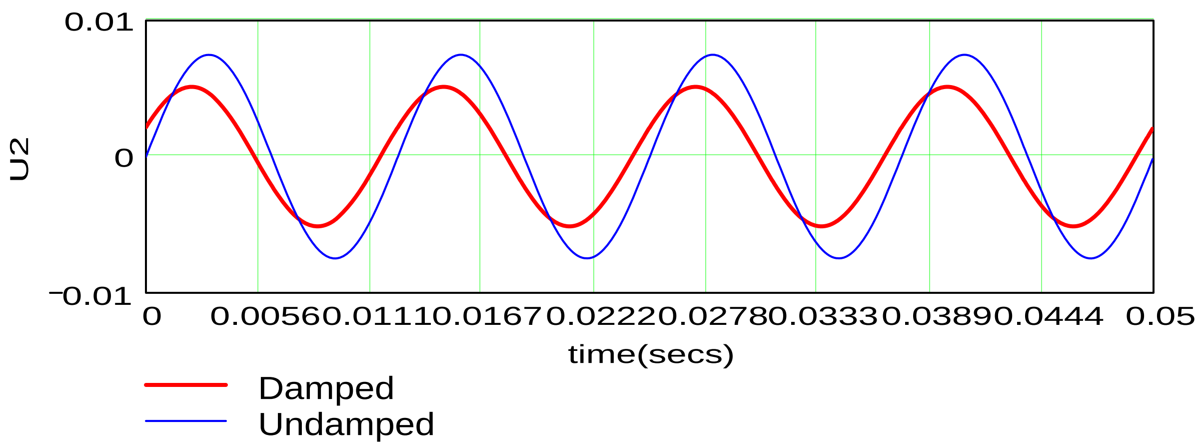
recall: $\Omega = 502.65$

$$\begin{pmatrix} 165.46 \\ 475.33 \\ 804.17 \end{pmatrix}$$

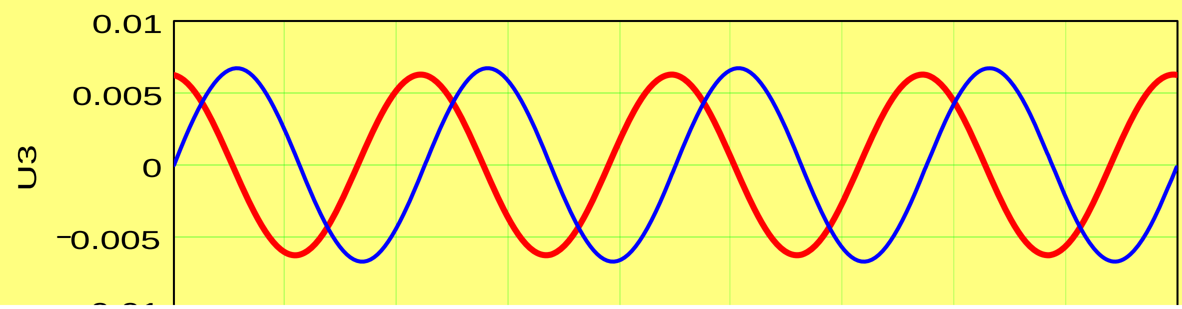
RESPONSE: U1



RESPONSE: U2



RESPONSE: U3



6) FRF Response to periodic loading, F=Fo cos(Ωt)

UNDAMPED CASE

SET $\xi := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$i := 1 .. n$

$Fo = \begin{pmatrix} 3 \times 10^4 \\ 1 \times 10^5 \\ -1 \times 10^5 \end{pmatrix}$

modal force magntidue: $P := \Phi^T \cdot Fo$

$\omega_i = \begin{pmatrix} 165.46 \\ 475.33 \\ 804.17 \end{pmatrix}$

and the modal "S-S" response is modes.

$j := 1 .. n$

$\Omega max := 3 \cdot \omega_n$

$kmin := 1$ $\quad kmax := 200$

$k := kmin .. kmax$

rad/sec <=== forcing frequency

$$\Omega_k := \frac{k}{kmax} \cdot \Omega max$$

$\Omega_{kmax} = 2.41 \times 10^3$

$\Omega_{kmin} = 12.06$

MODAL response

$$q_{j,k} := \frac{P_j}{Km_{j,j}} \cdot \frac{1}{\left[1.0 - \frac{(\Omega_k)^2}{(\omega_j)^2}\right] + \left(2 \cdot \xi_j \cdot \frac{\Omega_k}{\omega_j}\right) \cdot i} \times \cos(\Omega t)$$

here i=imaginary unit

$s := 1 .. n$

FRF in physical plane:

$$U_{s,k} := \sum_{i=1}^{n} \Phi_{s,i} \cdot q_{i,k} \times \cos(\Omega t)$$

DAMPED CASE

set: $Qo := stack(O, Fo)$

forcing vector $Go := \Phi_D{}^T \cdot Qo$

$k := kmin .. kmax$

$$\Omega_k := \frac{k}{kmax} \cdot \Omega max$$

rad/s <=== forcing frequency

$j := 1 .. 2 \cdot n$

modal response

$$Z_{j,k} := \frac{Go_j}{\beta_{j,j}} \cdot \frac{1}{\left(1 - i \cdot \frac{\Omega_k}{\alpha_j}\right)} \times \cos(\Omega t)$$

and back into the physical plane:

$s := 1 .. 2 \cdot n \qquad m := 2 \cdot n$

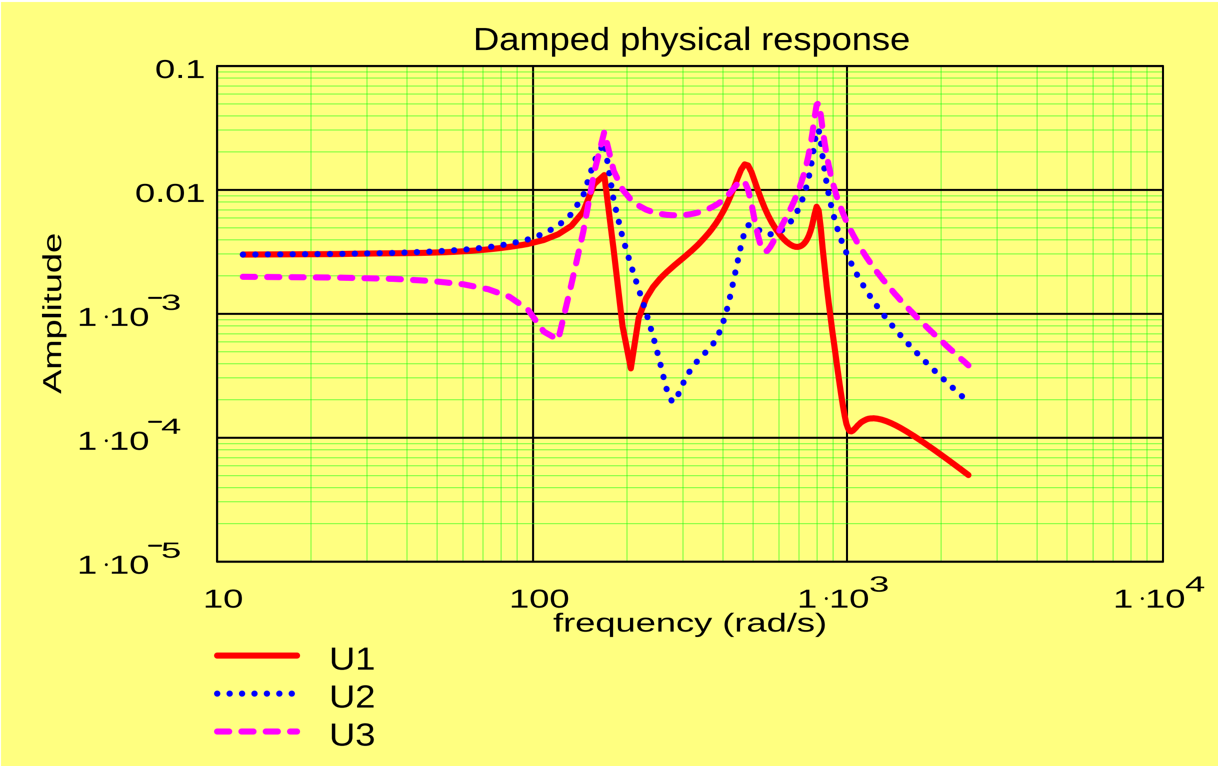
PERIODIC RESPONSE
using DAMPED MODES

$$Y_{s,k} := \sum_{q=1}^{m} \Phi D_{s,q} \cdot Z_{q,k}$$

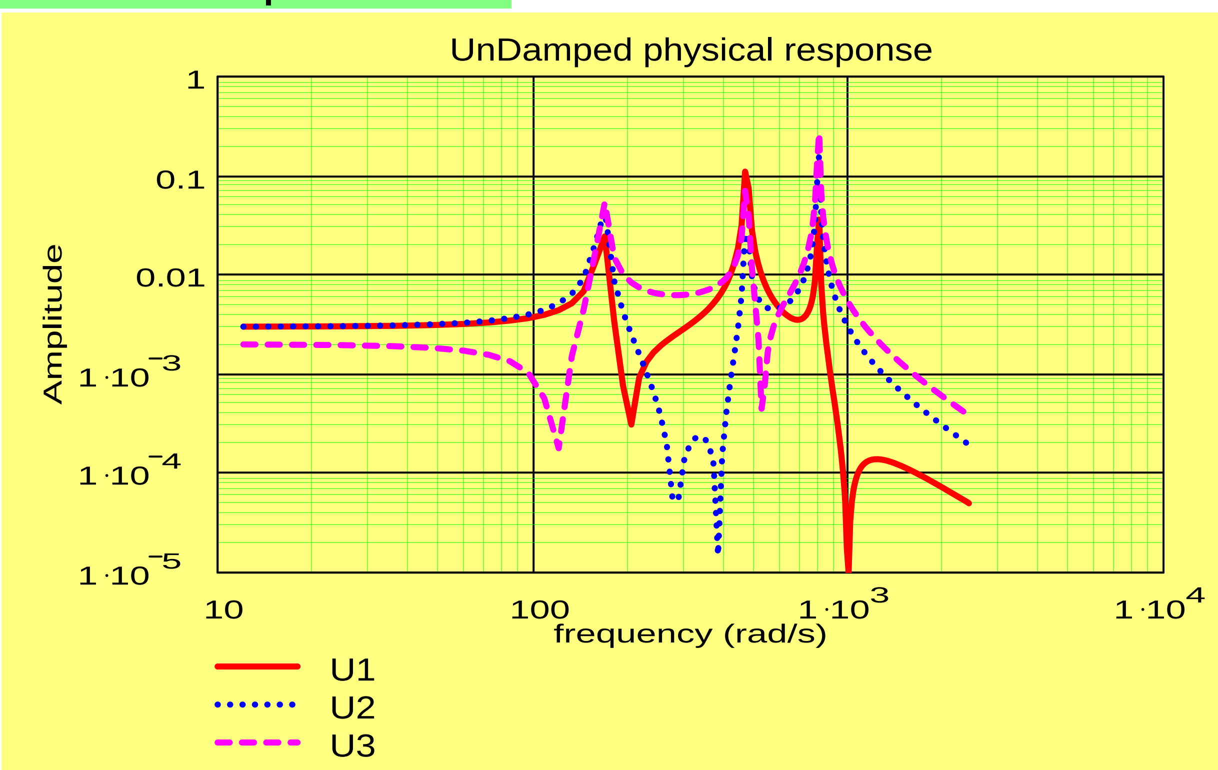
x cos(Ωt)

Plot **magnitude of response in physical space**

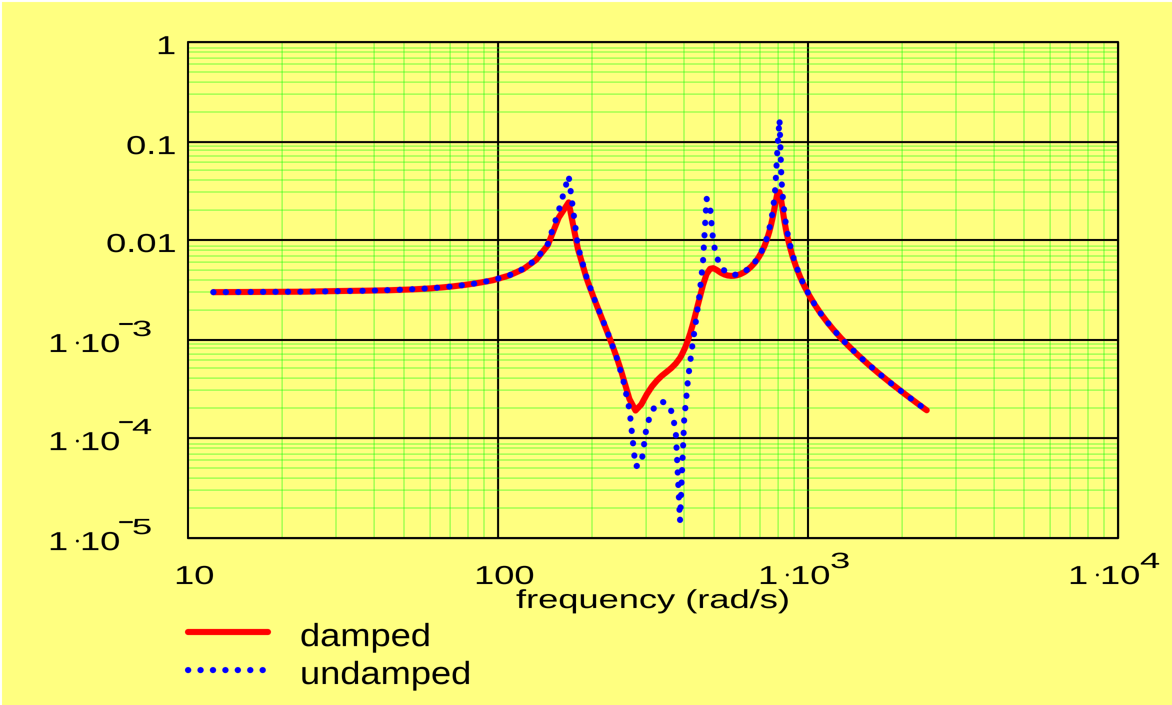
DAMPED Amplitude FRF



UNDAMPED Amplitude FRF



select coordinate to displate physical response - damped & undamped $jj := 2$ $n = 3$ **DOFs**



LINEAR vertical scale **undamped:** $\omega^T = (165.46 \quad 475.33 \quad 804.17)$