Lectures 22-23

Date: April 4 2017

Today: **Vibrations of continuous systems**

HD#14 Dynamic response of continuous systems

Free vibrations of elastic bars and beams.

Properties of normal mode functions. Forced response

Reading & other assignments:

Textbook G: 7.1 - 7.7, Handout #12

FE analysis

Other: HMWK #6: due 04/11

Handout 6 - Numerical integration (non linear systems)

From your textbook

Chapters 6&7: Vibration of elastic bars, beams and rods

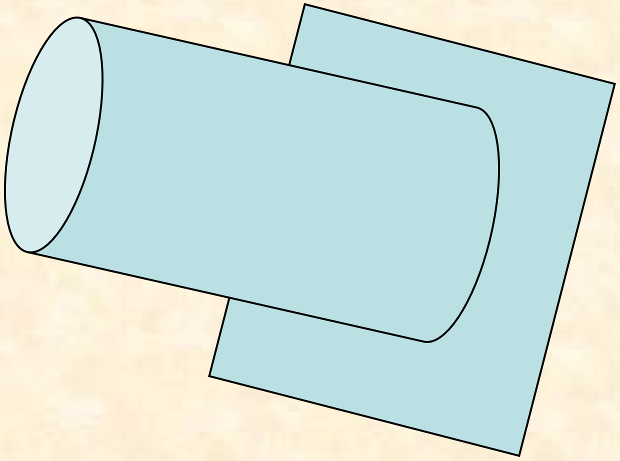
Recommended problems –

Chapter 6

3, 9, 11, 14, 15, 28,38, 54

Chapter 7

3,11,43,49



important

Vibrations of Continuous Systems

Axial vibrations of elastic bars

The figure shows a uniform elastic bar of length *L* and cross section *A*. The bar material properties are its density ρ and elastic modulus *E*. One end of the bar is attached to a fixed wall while the other end is free. The force *P(t)* acting at the free end of the bar induces elastic displacements *u(x,t)* along the bar

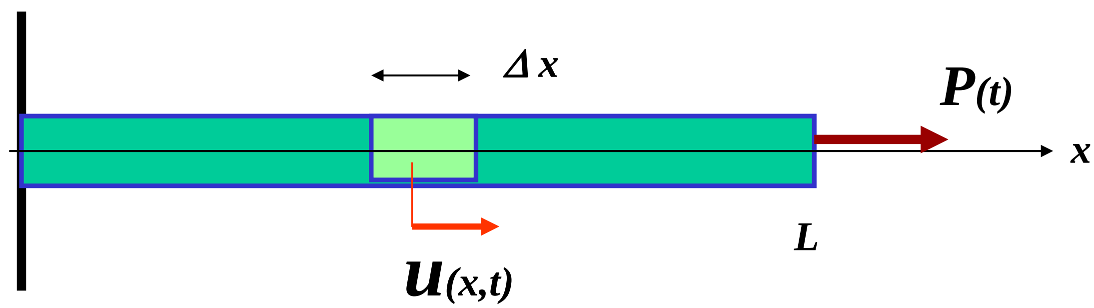


Fig. Schematic view of elastic bar undergoing axial motions

From elementary strength of materials consider

a) Cross-sections *A* remain plane and perpendicular to the main axis (*x*) of the bar.

b) Material is linearly elastic

c) Material properties (ρ, E) are constant at any given cross section.

The relationship between stress σ and strain ε for uniaxial tension is

$$\sigma = E\,\varepsilon = E\frac{\partial u}{\partial x} \qquad (1)$$

Consider the free body diagram of an infinitesimally small piece of bar with length Δx,

In the FBD, $P(x,t) = A_{(x)}\,\sigma = A\,E\,\frac{\partial u}{\partial x}$ is the axial force at a cross section of the bar, and $f(x,t)$ is a distributed axial force per unit length,

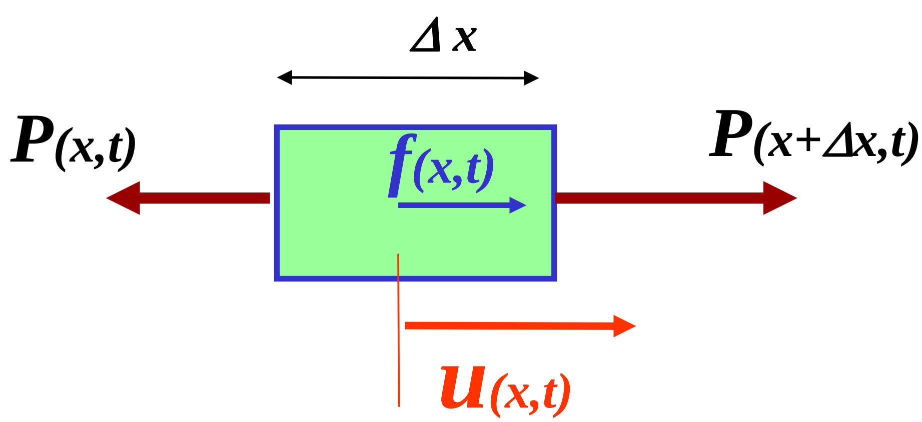


Fig. Free body diagram of small piece of elastic bar

Applying Newton's 2nd law of motion on the bar differential element gives

$$\sum_x F_x = \Delta m\, a_x = (\rho\, A\, \Delta x)\frac{\partial^2 u}{\partial t^2} \qquad (2)$$

$$(\rho\, A\, \Delta x)\frac{\partial^2 u}{\partial t^2} = P_{(x+\Delta x,t)} - P_{(x,t)} + f_{(x,t)}\Delta x \qquad (3)$$

As $\Delta x \to 0 \;\; \Rightarrow P_{(x+\Delta x,t)} \approx P_{(x,t)} + \frac{\partial P}{\partial x}\Delta x$ (4a)

$$\left(\rho A \Delta x\right)\frac{\partial^2 u}{\partial t^2}=\frac{\partial P}{\partial x}\Delta x+f_{(x,t)}\Delta x$$

$$\rho A\frac{\partial^2 u}{\partial t^2}=\frac{\partial P}{\partial x}+f_{(x,t)} \tag{4}$$

And replacing $P(x,t)=AE\frac{\partial u}{\partial x}$

$$\rho A\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right)+f_{(x,t)} \tag{5}$$

PDE (5) describes the axial motions of an elastic bar. For its solution, one needs appropriate boundary conditions *(BC)*, which are of two types

(a) **essential**, $u=u_*$, a specified value, at $x=x_*$ for all times,

(b) **natural,** $P(x_*,t) = \left. AE\frac{\partial u}{\partial x}\right)_{x=x_*}$ specified

If ***P=0***, then the natural BC is a **free end**, i.e. $\left.\frac{\partial u}{\partial x}\right)_{x=x_*}=0$

Note: PDE (5) and its BCs can be derived from the Hamiltonian principle using the definitions for kinetic (*T)* and potential (*V)* energies.

$$T=\frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 dx; \quad V=\frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 dx \tag{6}$$

recommended exercise (bonus) +5 to exam 2

Free vibrations of elastic bars

Without external forces (point loads or distributed load, $f=0$), PDE (5) reduces to

$$\rho A \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(A E \frac{\partial u}{\partial x} \right) \qquad (7)$$

METHOD: Separation of variables

The solution of PDE (7) is of the form $u_{(x,t)} = \phi_{(x)} v_{(t)}$ (8)

Note that

$$\frac{\partial^2 u}{\partial t^2} = \phi_{(x)} \frac{d^2 v}{d t^2} = \phi_{(x)} \ddot{v}_{(t)} \; ;$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 \phi}{d x^2} v_{(t)} = \phi'' v_{(t)} \qquad (9)$$

With the definitions $\left(\dot{} \right) = d/_{dt}$; $\left(' \right) = d/_{dx}$. For a bar with uniform material properties (ρ, E) and cross section A, substitution of the product solution Eq. (8) into PDE (7) gives

$$\frac{\rho}{E} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \;\rightarrow\; \frac{\rho}{E} \phi_{(x)} \ddot{v}_{(t)} = \phi''_{(x)} v_{(t)} \qquad (10)$$

Divide this expression by $u_{(x,t)} = \phi_{(x)} v_{(t)}$ to get

$$\frac{\ddot{v}_{(t)}}{v_{(t)}} = \frac{E}{\rho} \frac{\phi''_{(x)}}{\phi_{(x)}} \qquad (11)$$

Above, the *LHS* is only a function of time, while the *RHS* is only a function of spatial coordinate *x*. This is possible only if both sides equal to a constant, i.e.

$$\frac{\ddot{v}_{(t)}}{v_{(t)}} = \frac{E}{\rho}\frac{\phi''_{(x)}}{\phi_{(x)}} = -\omega^2$$

Hence, the PDE is converted into two ordinary differential equations (ODEs), i.e.

$$\ddot{v}_{(t)} + \omega^2 v = 0$$
$$\phi''_{(x)} + \lambda^2 \phi_{(x)} = 0 \qquad (12)$$

where $$\lambda = \omega\sqrt{\rho/E} \qquad (13)$$

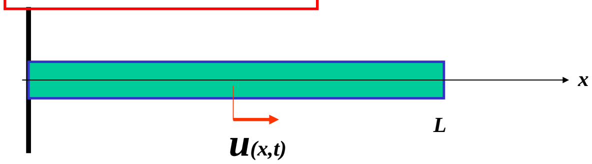
The solution of the ODEs (12) & (13) is

$$v_{(t)} = C_t \cos(\omega t) + S_t \sin(\omega t) \qquad (14)$$

$$\phi_{(x)} = C_x \cos(\lambda x) + S_x \sin(\lambda x) \qquad (15)$$

The coefficients (*C, S*) are determined from satisfying the boundary conditions for the specific bar configuration and load condition. Equation (15) is known as the **fundamental equation** for an elastic bar, i.e. it contains the information on natural frequencies and mode shapes.

Example 1.



A bar with one end fixed and the other end free.
In this case, the boundary conditions are

for all times

At $x=0$, $$u_{(0,t)}=0=\phi_{(0)}v_{(t)} \Rightarrow \phi_{(0)}=0 \quad \forall t$$

At $x=L$, $$\left.\frac{\partial u}{\partial x}\right]_{x=L}=0=\phi'_{(L)}v_{(t)} \Rightarrow \phi'_{(L)}=0 \quad \forall t \qquad (16)$$

Hence, from the characteristic equation $\phi_{(0)}=0 \rightarrow C_x=0$ and

$$\phi_{(x)}=S_x \sin(\lambda x) \qquad (17)$$

At $x=L$, $$\phi'_{(L)}=0=\lambda S_x \cos(\lambda L)=0 \qquad (18)$$

Note that $S_x \neq 0$ for a non trivial solution. Hence, the **characteristic equation** for axial motions of a **fixed end-free end elastic bar** is

$$\cos(\lambda L)=0 \qquad (19)$$

which has an infinite number of solutions, i.e.

$$\lambda L=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},....,\infty=\frac{2n-1}{2}\pi, \quad n=1,2,.....$$

And hence the roots of Eq. (19) are

$$\lambda_n=\frac{(2n-1)}{2}\frac{\pi}{L} \qquad n=1,2,..... \qquad (20)$$

And since $\lambda = \omega\sqrt{\rho/E}$, the natural frequencies of the fixed end-free end bar are

$$\omega_k = \frac{(2k-1)}{2}\frac{\pi}{L}\left(\frac{E}{\rho}\right)^{1/2} \quad ; \quad _{k=1,2,.....} \tag{21}$$

i.e. $\omega_1 = \frac{\pi}{2L}\left(\frac{E}{\rho}\right)^{1/2}$, $\omega_2 = \frac{3\pi}{2L}\left(\frac{E}{\rho}\right)^{1/2}$, $\omega_3 = \frac{5\pi}{2L}\left(\frac{E}{\rho}\right)^{1/2}$ ....

Associated to each natural frequency, there is a **natural mode shape**

$$\phi_k = \psi_k = \sin(\lambda_k x) \quad _{k=1,2,.....} \tag{22}$$

as shown in the figure below.

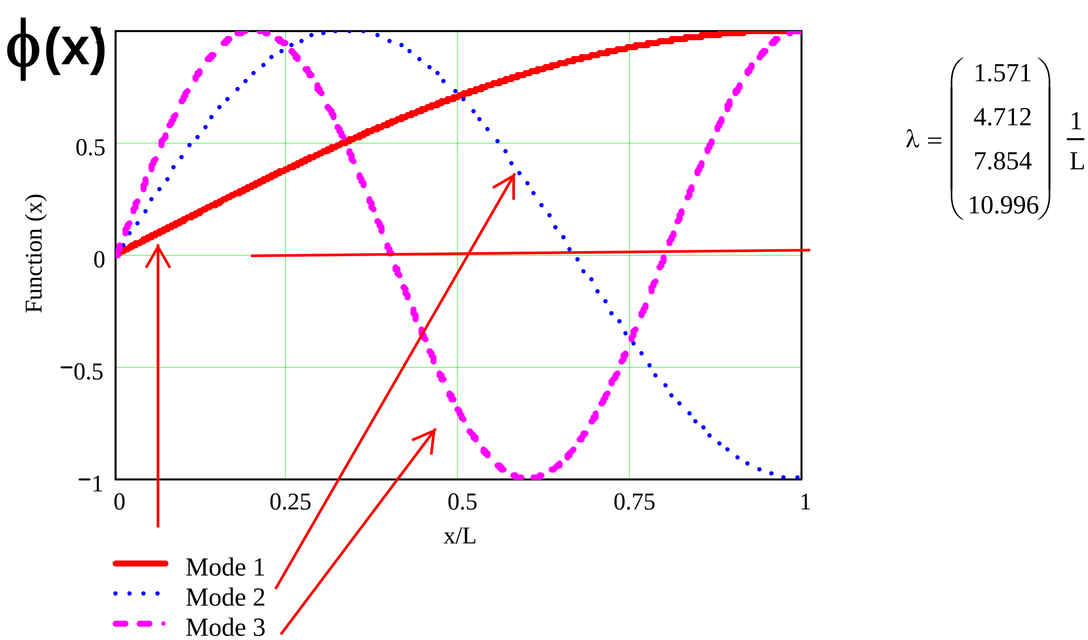


Fig. Natural modes shapes ϕ(x) for elastic bar with fixed end-free end

See more examples on page 13-ff.

The **displacement function response** $u_{(x,t)} = \phi_{(x)} v_{(t)}$ equals to the superposition of all the found responses, i.e.

$$u_{(x,t)} = \sum_k \phi(x)_k \, v(t)_k =$$

$$\sum_{k=1}^{\infty} \phi_{(x)_k} \left[C_k \cos(\omega_k t) + S_k \sin(\omega_k t) \right] \tag{23a}$$

For example 1 (fixed end –free end bar)

$$u_{(x,t)} = \sum_{k=1}^{\infty} \sin(\lambda_k x) \left[C_k \cos(\omega_k t) + S_k \sin(\omega_k t) \right] \tag{23b}$$

and velocity:

$$\dot{u}_{(x,t)} = \sum_{k=1}^{\infty} \sin(\lambda_k x) \, \omega_k \left[-C_k \sin(\omega_k t) + S_k \cos(\omega_k t) \right] \tag{24}$$

The set of coefficients (C_k, S_k) are determined by satisfying the initial conditions. That is at time $t=0$,

$$u_{(x,0)} = U_{(x)} = \sum_{k=1}^{\infty} \sin(\lambda_k x) C_k$$

$$\dot{u}_{(x,0)} = \dot{U}_{(x)} = \sum_{k=1}^{\infty} \omega_k \sin(\lambda_k x) S_k \tag{25}$$

Orthogonality properties of the natural modes

Recall that the pair $\left\{\lambda_k, \psi_{(x)_k}\right\}_{k=1,\ldots\infty}$ satisfy the characteristic equation (12b), i.e.

$$\psi''_{(x)_k} + \lambda_k^2 \psi_{(x)_k} = 0 \qquad k=1,2\ldots\infty \tag{26}$$

And consider two **different** eigenvalues λ_i and λ_j each satisfying Eq. (26), i.e.

$$\psi''_i + \lambda_i^2 \psi_i = 0 \quad \& \quad \psi''_j + \lambda_j^2 \psi_j = 0$$

Multiply Eq. on left by ψ_j and Eq. on right by ψ_i, and integrate over the domain $x \in \{0, L\}$ to get:

$$\int_0^L \left(\psi_j \psi''_i \, dx\right) + \lambda_i^2 \int_0^L \left(\psi_j \psi_i \, dx\right) = 0$$

$$\int_0^L \left(\psi_i \psi''_j \, dx\right) + \lambda_j^2 \int_0^L \left(\psi_i \psi_j \, dx\right) = 0 \tag{27}$$

Integrate by parts the term on the LHS to obtain

INT(u dv) = uv - INT (v du)

$$\int_0^L \psi_j \psi''_i \, dx = \left(\psi_j \psi'_i\right]_{x=0}^{x=L} - \int_0^L \psi'_j \psi'_i \, dx \tag{28}$$

And recall the boundary conditions for the **fixed end-free end bar**

$$\left(\psi_j\right]_{x=0} = 0 \quad \& \quad \left(\psi'_i\right]_{x=L} = 0 \tag{29}$$

And write first of Eq. (27) as $\lambda_i^2 \int_0^L (\psi_j \psi_i \, dx) = \int_0^L (\psi'_j \psi'_i \, dx)$ and substituting $\lambda_i = \omega_i \sqrt{\rho / E}$ one obtains:

$$\omega_i^2 \int_0^L (\rho A \psi_j \psi_i \, dx) = \int_0^L (E A \psi'_j \psi'_i \, dx) \qquad (30a)$$

$$\omega_j^2 \int_0^L (\rho A \psi_i \psi_j \, dx) = \int_0^L (E A \psi'_i \psi'_j \, dx) \qquad (30b)$$

Subtract Eq. (30b) from (30a) to obtain

$$\left(\omega_j^2 - \omega_i^2\right) \int_0^L (\rho A \psi_i \psi_j) \, dx = 0 \qquad (31)$$

And since $\omega_i \neq \omega_j$, it follows that

$$\int_0^L (\rho A \psi_i \psi_j) \, dx = 0 \;\; \& \;\; \int_0^L (E A \psi'_i \psi'_j) \, dx = 0 \quad {}_{i \neq j = 1,2,\ldots\infty} \qquad (32)$$

That is, the modal functions $\{\psi_k\}_{k=1,2\ldots}$ are **ORTHOGONAL.** For $i=j$, the i_{th} natural frequency follows from

$$\omega_i^2 = \frac{K_i}{M_i} = \frac{\int_0^L (E A \psi'_i \psi'_i) \, dx}{\int_0^L (\rho A \psi_i \psi_i) \, dx} \qquad (33)$$

Where K_i, M_i are the i_{th} mode *equivalent* stiffness and mass coefficients.

Note that the set $\{\psi_k\}_{k=1,2...}$ is a **COMPLETE SET of** orthogonal functions

Now, consider the initial conditions, Eq. (25)

$$u_{(x,0)} = U_{(x)} = \sum_{k=1}^{\infty} \sin(\lambda_k x) C_k$$
$$\dot{u}_{(x,0)} = \dot{U}_{(x)} = \sum_{k=1}^{\infty} \omega_k \sin(\lambda_k x) S_k \qquad (25)$$

Multiply both sides of Eq. (25) by $\psi_m = \sin(\lambda_m x)$ x ρA and integrate over the whole domain to obtain

$$\int_0^L \left(\rho A \psi_m U_{(x)}\right) dx = \sum_{k=1}^{\infty} C_k \int_0^L \left(\rho A \psi_m \psi_k\right) dx$$

And since

$$\int_0^L \left(\rho A \psi_m \psi_k\right) dx = \begin{Bmatrix} \mathrm{M}_m & \text{when } m=k \\ 0 & \text{when } m \neq k \end{Bmatrix} \qquad (34)$$

Then if follow that

$$C_m = \frac{\int_0^L \left(\rho A \psi_m U_{(x)}\right) dx}{\mathrm{M}_m}, \quad m=1,2,...\infty \qquad (35)$$

And similarly

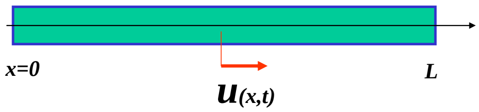
$$S_m = \frac{\int_0^L \left(\rho A \psi_m \dot{U}_{(x)}\right) dx}{\omega_m \mathrm{M}_m}, \quad m=1,2,...\infty \qquad (36)$$

with $M_m = \int_0^L \left(\rho A \psi_m^2\right) dx$ and $K_m = \int_0^L \left(E A \left[{}^{d\psi_m}\!/_{dx} \right]^2 \right) dx$ (37)

This concludes the procedure to obtain the full solution for the vibrations of a bar, i.e.

$$u_{(x,t)} = \sum_{k=1}^{\infty} \phi_{(x)_k} \left[C_k \cos(\omega_k t) + S_k \sin\left(\omega_k t\right) \right] \quad (23)$$

Example 2.



A bar with both ends free.
The boundary conditions are

At $x=0$, $$\left.\frac{\partial u}{\partial x}\right]_{x=0} = 0 = \phi'_{(0)} v_{(t)} \Rightarrow \quad \phi'_{(0)} = 0 \quad \forall t$$

At $x=L$, $$\left.\frac{\partial u}{\partial x}\right]_{x=L} = 0 = \phi'_{(L)} v_{(t)} \Rightarrow \quad \phi'_{(L)} = 0 \quad \forall t$$

Hence, from the characteristic equation $\phi'_{(0)} = 0 \rightarrow S_x = 0$ and

$$\phi_{(x)} = C_x \cos(\lambda x)$$

At $x=L$, $$\phi'_{(L)} = 0 = \lambda C_x \sin(\lambda L) = 0$$

Note that $\lambda = 0$ denotes rigid body motion. Hence, the **characteristic equation** for axial motions of an **elastic bar** with **free-free ends** is

$$\sin(\lambda L) = 0$$

which has an infinite number of solutions, i.e.

$$\lambda L = 0, \pi, 2\pi, 3\pi, ..., \infty = n\pi, \quad {}_{n=0,1,2.....}$$

$$\lambda_n = n\frac{\pi}{L} \qquad {}_{n=0,1,2,.....}$$

And since $\lambda = \omega\sqrt{\rho/E}$, the natural frequencies of the free end-free end bar are

$$\omega_k = k\frac{\pi}{L}\left(\frac{E}{\rho}\right)^{1/2} \quad ; \quad {}_{k=0,1,2,.....}$$

Associated to each natural frequency, there is a **natural mode shape**

$$\phi_k = \cos(\lambda_k x) \quad {}_{k=0,1,2,...}$$

And shown in the figure below.

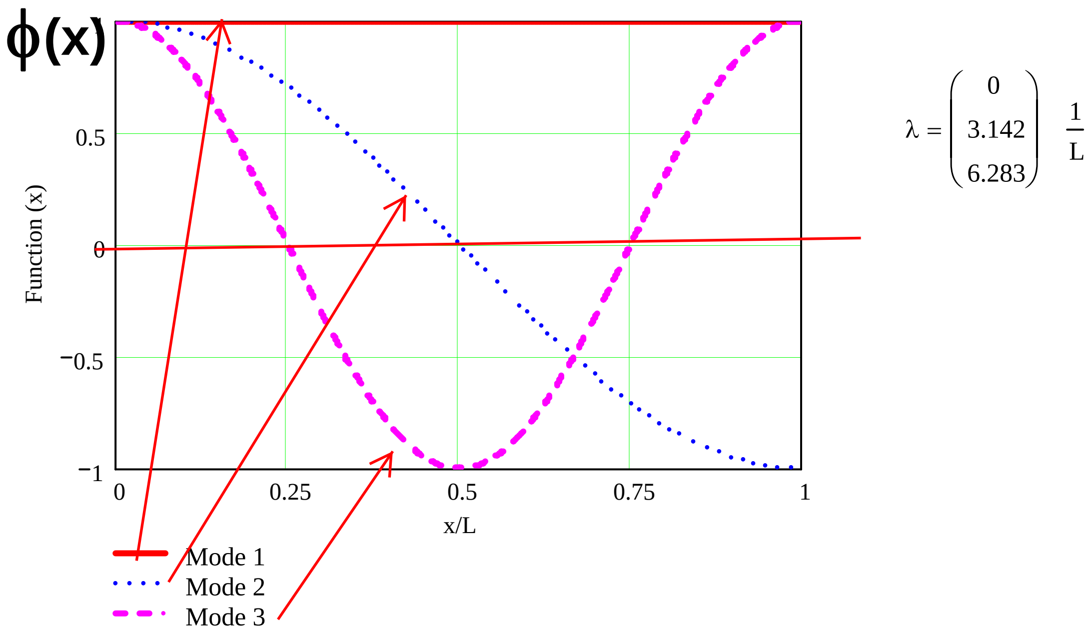
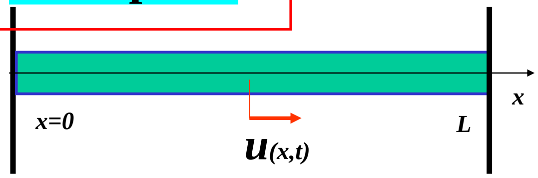


Fig. Natural modes shapes ϕ(x) for elastic bar with both ends free. First mode is rigid body (null natural frequency)

Example 3.



A bar with both ends fixed. The boundary conditions are

At $x=0$, $u_{(0,t)}=0=\phi_{(0)}v_{(t)} \Rightarrow \phi_{(0)}=0 \quad \forall t$

At $x=L$, $u_{(L,t)}=0=\phi_{(L)}v_{(t)} \Rightarrow \phi_{(L)}=0$

Hence, from the characteristic equation $\phi_{(x)}=C_x\cos(\lambda x)+S_x\sin(\lambda x)$, then $\phi_{(0)}=0 \rightarrow C_x=0$ and

$$\phi_{(x)}= S_x\sin(\lambda x)$$

At $x=L$, $\phi_{(L)}=0=\sin(\lambda L)=0$

Note that $\lambda \neq 0$ denotes rigid body motion. Hence, the **characteristic equation** for axial motions of a **fixed end-fixed end elastic bar** is

$$\sin(\lambda L)=0$$

which has an infinite number of solutions, i.e.

$$\lambda L=\pi, 2\pi, 3\pi, ..., \infty = n\pi, \quad {}_{n=0,1,2.....}$$

$$\lambda_n = n\frac{\pi}{L} \qquad {}_{n=1,2,.....}$$

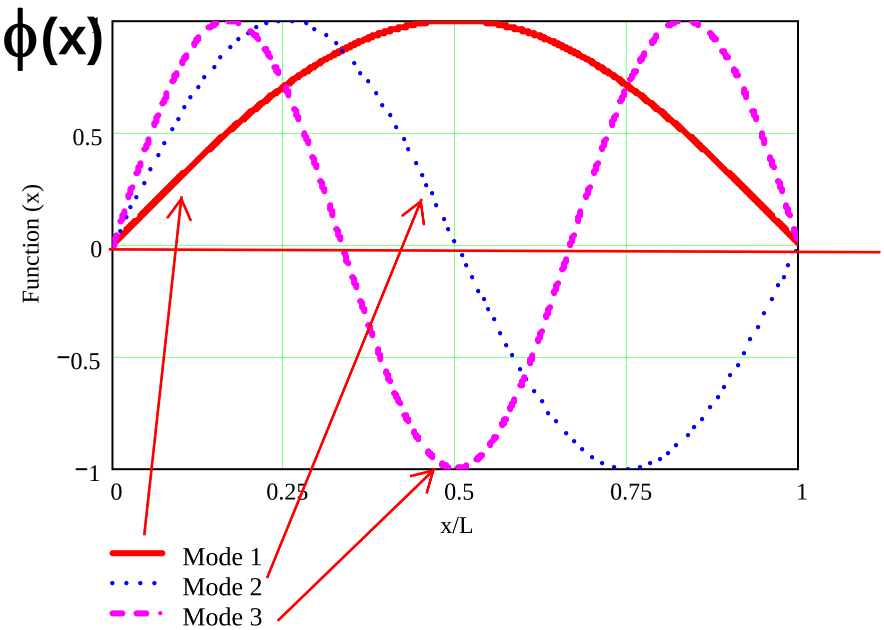
And since $\lambda=\omega\sqrt{\rho/E}$, the natural frequencies of the free end-free end bar are

$$\omega_k = k\frac{\pi}{L}\left(\frac{E}{\rho}\right)^{1/2} \quad ; \quad {}_{k=1,2,.....}$$

Associated to each natural frequency, there is a **natural mode shape**

$$\phi_k = \sin(\lambda_k x) \quad {}_{k=0,1,2,...}$$

And shown in the figure below.



$$\lambda = \begin{pmatrix} 3.142 \\ 6.283 \\ 9.425 \end{pmatrix} \frac{1}{L}$$

Fig. Natural modes shapes ϕ(x) for elastic bar with both ends fixed.

Lateral vibrations of elastic beams

The figure shows a uniform elastic beam of length *L,* cross section *A* and area moment of inertia *I.* The beam material properties are its density ρ and elastic modulus *E.* One end of the beam is fixed to a wall while the other end is free. The discrete force $P(t)$ acts at a fixed axial location while $f(x,t)$ represents a load distribution per unit length. The forces induces elastic displacements on the beam and designated as $v(x,t)$.

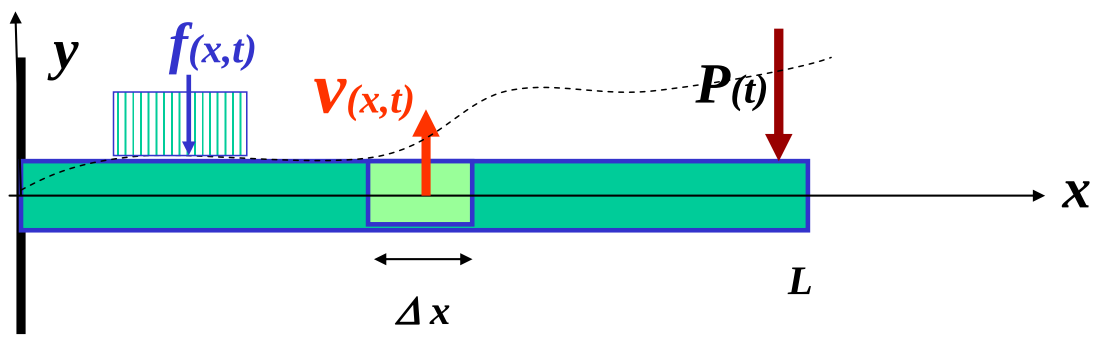


Fig. Schematic view of elastic beam undergoing lateral motions

From elementary strength of materials consider

a) Cross-sections *A* remain plane and perpendicular to the neutral axis (x) of the beam.

b) Homogeneous material beam, linearly elastic,

c) Material properties (ρ, E) are constant at any given cross section.

d) Stresses σ_y , σ_z << σ_x (flexural stress), i.e. along beam.

The graph below shows the free body diagram for motion of a differential beam element with length Δx.

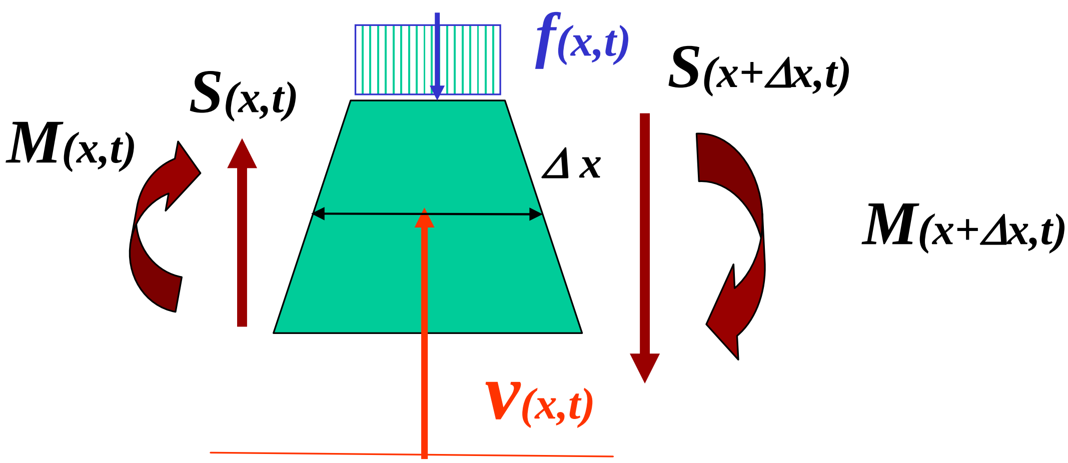


Fig. Free body diagram of small piece of elastic beam

In The *FBD,* $S_{(x,t)}$ represents the shear force and $M_{(x,t)}$ denotes the bending moment. Apply Newton's 2[nd] law to the material element:

$$\sum_x F_y = \Delta m\; a_y = S - \left(S + \frac{\partial S}{\partial x}\Delta x \right) + f_{(x,t)} = \left(\rho\, A\, \Delta x\right) \frac{\partial^2 v}{\partial t^2} \qquad (38)$$

In the limit as $\Delta x \to 0$:

$$\left(\rho\, A\right) \frac{\partial^2 v}{\partial t^2} = f_{(x,t)} - \frac{\partial S}{\partial x} \qquad (39)$$

Apply the moment equation: $\sum M = \Delta I_g\, \ddot{\alpha} \sim 0$ (40)

neglecting rotary inertia ΔI_g

Then $$\sum M \approx 0 = M_{(x+\Delta x,t)} - M_{(x,t)} - f\frac{\Delta x^2}{2} - S\,\Delta x$$

$$= M + \frac{\partial M}{\partial x}\Delta x - M - f\frac{\Delta x^2}{2} - S\,\Delta x$$

In the limit as $\Delta x \to 0$:

$$\frac{\partial M}{\partial x} = S_{(x,t)} \qquad (41)$$

Combining Eqs. (41) and (39) gives:

$$(\rho A)\frac{\partial^2 v}{\partial t^2} = f_{(x,t)} - \frac{\partial^2 M}{\partial x^2} \qquad (42)$$

If the slope $\left(\partial v/\partial x\right)$ remains small, then the beam curvature is $1/\tilde{\rho} = \partial^2 v/\partial x^2$. From **Euler's beam theory**:

$$M = \frac{EI}{\tilde{\rho}} = EI\frac{\partial^2 v}{\partial x^2} \qquad (43)$$

where $I = \iint \rho\, y^2\, dA$ is the beam second moment of area (m^4)

Substitute Eq. (43) into (42) to obtain the **equation for lateral motions of an elastic beam**:

$$(\rho A)\frac{\partial^2 v}{\partial t^2} = f_{(x,t)} - \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x^2}\right) \qquad (44)$$

The PDE is fourth-order in space and 2nd order in time. Appropriate boundary conditions are of two types:

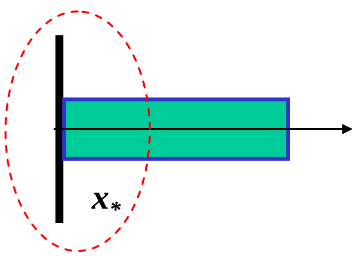
Essential BCs:

- specified displacement, $v = v_*$
- specified slope, $\left({}^{\partial v}\!/\!{}_{\partial x} \right) = \theta_*$

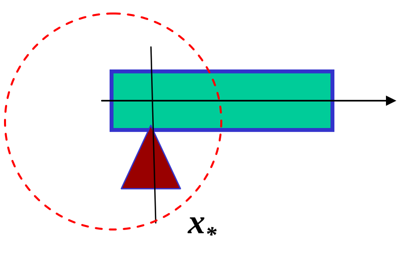
Natural BCs:

- specified moment, $M = M_* = E\,I \left(\dfrac{\partial^2 v}{\partial x^2} \right)_{x_*}$
- specified shear force, $S = S_* = \dfrac{\partial}{\partial x} \left(E\,I \dfrac{\partial^2 v}{\partial x^2} \right)_{x_*}$

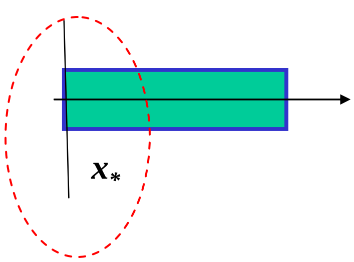
See below the most typical beam configurations:



Fixed end (cantilever): $v = 0 \ \& \ \dfrac{\partial v}{\partial x} = 0$

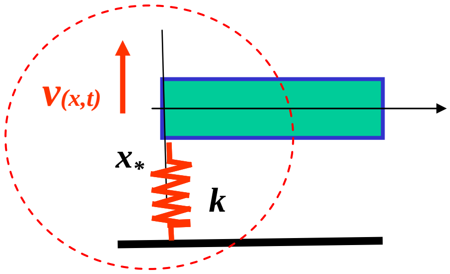


Pinned end $v = 0 \ \& M = 0 \rightarrow \dfrac{\partial^2 v}{\partial x^2} = 0$



Free end

$$M = 0 \ \& S = 0 \rightarrow \frac{\partial^2 v}{\partial x^2} = 0 \ \& \ \frac{\partial^3 v}{\partial x^3} = 0$$



Spring supported end

$$M = \frac{\partial^2 v}{\partial x^2} = 0$$

$$-S = k\, v_* = -\frac{\partial}{\partial x}\left(E I \frac{\partial^2 v}{\partial x^2} \right)_{x_*}$$

Note: PDE (44) and its BCs can be derived from the Hamiltonian principle using the definitions for kinetic (*T*) and potential (*V*) energies of an elastic beam

A rewarding intellectual pursuit, Bonus +5 to Exam 2 (by Tuesday 04/11)

$$T = \frac{1}{2}\int_0^L \rho A \left(\frac{\partial v}{\partial t} \right)^2 dx; \quad V = \frac{1}{2}\int_0^L E I \left(\frac{\partial^2 v}{\partial x^2} \right)^2 dx$$

Free vibrations of elastic beam

Without external forces (point loads or distributed load, *f=0*), PDE (44) reduces to

$$(\rho A)\frac{\partial^2 v}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left(E I \frac{\partial^2 v}{\partial x^2} \right) \qquad (46)$$

The solution of PDE (46) is of the form $v_{(x,t)} = \phi_{(x)} \mathrm{v}_{(t)}$ (47)

Let $(\dot{\ }) = {}^{d}/_{dt}; \ (\ ') = {}^{d}/_{dx}$. Substituting Eq. (47) into Eq (46) gives

$$\phi_{(x)} \ddot{\mathrm{v}}_{(t)} = \frac{E I}{\rho A} \mathrm{v} \frac{d^4 \phi_{(x)}}{d x^4} \quad \Rightarrow \quad \frac{\ddot{\mathrm{v}}_{(t)}}{\mathrm{v}} = \frac{E I}{\rho A} \frac{1}{\phi_{(x)}} \frac{d^4 \phi_{(x)}}{d x^4} = -\omega^2$$

Above, the *LHS* is only a function of time, while the *RHS* is only a function of spatial coordinate *x*. This is possible only if both sides are equal to a constant, i.e. $\left(-\omega^2\right)$. Hence, the separation of variables gives two ordinary differential equations

$$\ddot{v}_{(t)} + \omega^2 v = 0 \quad \& \quad \frac{d^4 \phi}{d x^4} - \lambda^2 \phi = 0 \tag{48}$$

where
$$\lambda^2 = \omega^2 \left(\frac{\rho A}{E I}\right) \tag{49}$$

The solution of the ODEs is

$$v_{(t)} = C_t \cos(\omega t) + S_t \sin(\omega t) \tag{50}$$

$$\phi_{(x)} = C_1 \cos(\beta x) + C_2 \sin(\beta x) + C_3 \cosh(\beta x) + C_4 \sinh(\beta x)$$
(51)[1]

where
$$\beta = \lambda^{1/2} = \omega^{1/2} \left(\frac{\rho A}{E I}\right)^{1/4} \tag{52}$$

has units of [1/length].

The coefficients (*C, S*) are determined from satisfying the boundary conditions for the specific beam configuration. Equation (51) is known as the **fundamental mode shape** for an elastic beam, i.e., it contains the information on natural frequencies and mode shapes.

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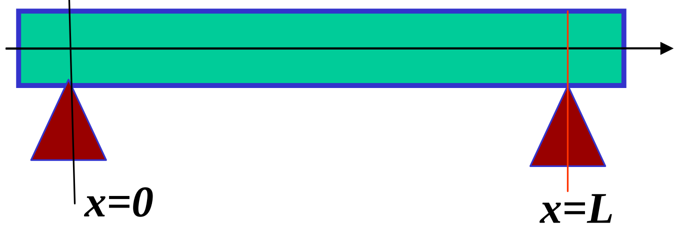
[1] The solution of ODE $\phi^{iv} - \lambda^2 \phi = 0 = \phi^{iv} - \beta^4 \phi = 0$ is $\phi = c\, e^{kx}$ with characteristic equation $k^4 - \lambda^2 = 0$

Example 1. Pin-pin ends beam



Recall $\phi_{(x)} = C_1\cos(\beta x) + C_2\sin(\beta x) + C_3\cosh(\beta x) + C_4\sinh(\beta x)$

The BCs are: for all time

At $x=0$, $v_{(0,t)} = 0 = \phi_{(0)} v_{(t)} \Rightarrow \phi_{(0)} = 0 \quad \forall t$ (53.a)

$\rightarrow \quad \phi_{(0)} = C_1 + C_3$

$M = \partial^2 v / \partial x^2 = 0 = \phi''_{(0)} v_{(t)} \Rightarrow \phi''_{(0)} = 0$

$\rightarrow \quad \phi''_{(0)} = -C_1 + C_3$

Hence, $C_1 = C_3 = 0$ and $\phi_{(x)} = C_2\sin(\beta x) + C_4\sinh(\beta x)$

At $x=L$, $v_{(L,t)} = 0 = \phi_{(L)} v_{(t)} \Rightarrow \phi_{(L)} = 0 \quad \forall t$

$\rightarrow \quad \phi_{(L)} = 0 = C_2\sin(\beta L) + C_4\sinh(\beta L)$

$M_{x=L} = \partial^2 v / \partial x^2 = 0 = \phi''_{(L)} v_{(t)} \Rightarrow \phi''_{(L)} = 0$ (53.b)

$\rightarrow \quad \phi''_{(L)} = 0 = -C_2\sin(\beta L) + C_4\sinh(\beta L)$

from this two equations, since $\sinh(\beta L) \neq 0$, it follows that

$$\phi_{(x)} = C_2\sin(\beta x) \quad (54)$$

where $\sin(\beta L) = 0$ when $\beta_i = \frac{i\pi}{L}, \quad {}_{i=1,2....\infty}$ (55)

and hence, the **natural frequencies of the pin-pin beam** are

$$\omega_i = \beta_i^2 \left(\frac{E\,I}{\rho\,A}\right)^{1/2} = \frac{i^2\,\pi^2}{L^2}\left(\frac{E\,I}{\rho\,A}\right)^{1/2} \; ; \quad i=1,2...\infty \qquad (56)$$

Associated to each natural frequency, there is a **natural mode shape**

$$\phi_i = \sin(\beta_i\,x) = \sin\left(\frac{i\,\pi\,x}{L}\right) ; \quad i=1,2,..... \qquad (57)$$

as shown in the graph below.

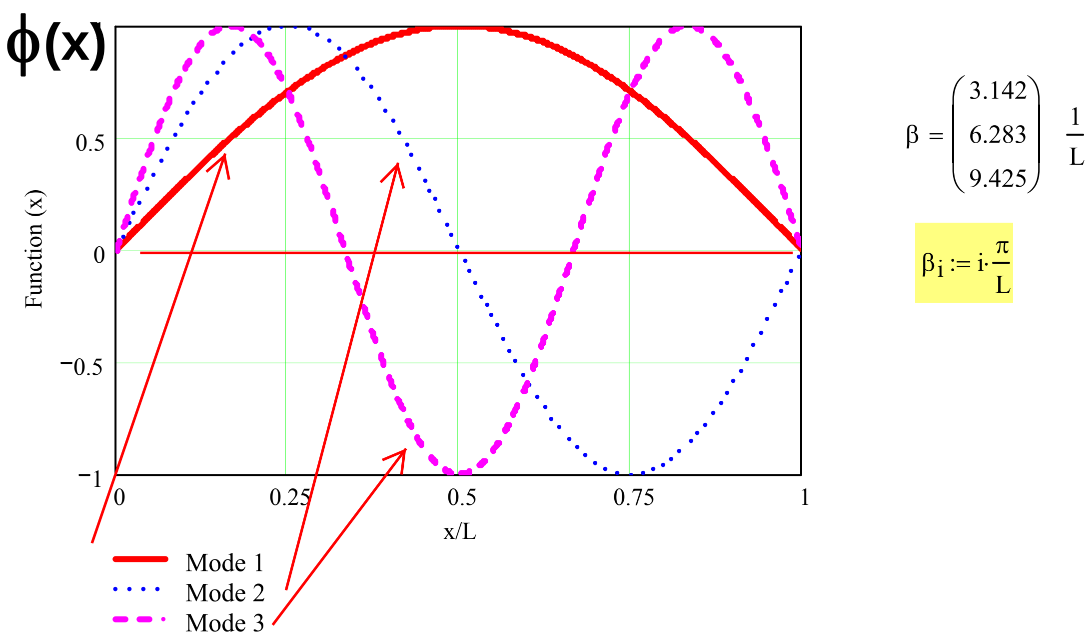


Fig. Natural mode shapes ϕ(x) for elastic beam with both ends pinned.

The displacement function response $v_{(x,t)} = \phi_{(x)} \mathrm{V}_{(t)}$ equals to the superposition of all the found responses, i.e.

$$v_{(x,t)} = \sum_k \phi(x)_k \, \mathrm{v}(t)_k = \sum_{k=1}^{\infty} \phi_{(x)_k} \left[C_k \cos(\omega_k t) + S_k \sin(\omega_k t) \right]$$

$$v_{(x,t)} = \sum_{k=1}^{\infty} \sin(\beta_k x) \left[C_k \cos(\omega_k t) + S_k \sin(\omega_k t) \right] \quad (58)$$

and velocity:

$$\dot{v}_{(x,t)} = \sum_{k=1}^{\infty} \sin(\beta_k x)\, \omega_k \left[-C_k \sin(\omega_k t) + S_k \cos(\omega_k t) \right] \quad (59)$$

The set of coefficients (C_k, S_k) are determined by satisfying the initial conditions. That is at time $t=0$,

$$v_{(x,0)} = V_{(x)} = \sum_{k=1}^{\infty} \sin(\beta_k x)\, C_k$$

$$\dot{v}_{(x,0)} = \dot{V}_{(x)} = \sum_{k=1}^{\infty} \omega_k \sin(\beta_k x)\, S_k \quad (60)$$

RECALL:

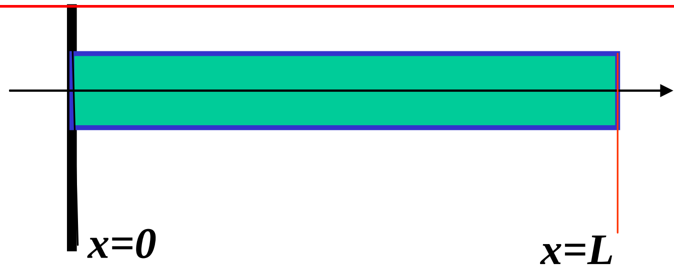
$$\phi_{(x)} = C_1 \cos(\beta x) + C_2 \sin(\beta x) + C_3 \cosh(\beta x) + C_4 \sinh(\beta x)$$

$$\phi' / \beta = -C_1 \sin(\beta x) + C_2 \cos(\beta x) + C_3 \sinh(\beta x) + C_4 \cosh(\beta x)$$

$$\phi'' / \beta^2 = -C_1 \cos(\beta x) - C_2 \sin(\beta x) + C_3 \cosh(\beta x) + C_4 \sinh(\beta x)$$

$$\phi''' / \beta^3 = C_1 \sin(\beta x) - C_2 \cos(\beta x) + C_3 \sinh(\beta x) + C_4 \cosh(\beta x)$$

Example 2. Fixed end-free end beam



Recall $\phi_{(x)} = C_1\cos(\beta x) + C_2\sin(\beta x) + C_3\cosh(\beta x) + C_4\sinh(\beta x)$

The BCs. are

At x=0, $\quad v_{(0,t)} = 0 = \phi_{(0)} \mathrm{v}_{(t)} \Rightarrow \quad \phi_{(0)} = 0 \quad \forall t \qquad$ (61.a)

$\rightarrow \qquad \phi_{(0)} = C_1 + C_3$

$\theta = {\partial v}/{\partial x} = 0 = \phi'_{(0)} \mathrm{v}_{(t)} \Rightarrow \quad \phi'_{(0)} = 0 \qquad$ (61.b)

$\rightarrow \qquad \phi'_{(0)} = C_2 + C_4$

At x=L

$M_{x=L} = {\partial^2 v}/{\partial x^2} = 0 = \phi''_{(L)} \mathrm{v}_{(t)} \Rightarrow \quad \phi''_{(L)} = 0 \qquad$ (61.c)

$\rightarrow$

$$\phi''_{(L)} = 0 = -C_1\cos(\beta L) - C_2\sin(\beta L) + C_3\cosh(\beta L) + C_4\sinh(\beta L)$$

$S_{x=L} = {\partial^3 v}/{\partial x^3} = 0 = \phi'''_{(L)} \mathrm{v}_{(t)} \Rightarrow \quad \phi'''_{(L)} = 0 \qquad$ (61.d)

$\rightarrow$

$$\phi'''_{(L)} = 0 = C_1\sin(\beta L) - C_2\cos(\beta L) + C_3\sinh(\beta L) + C_4\cosh(\beta L)$$

Solution of Eqs. (a)-(d) gives

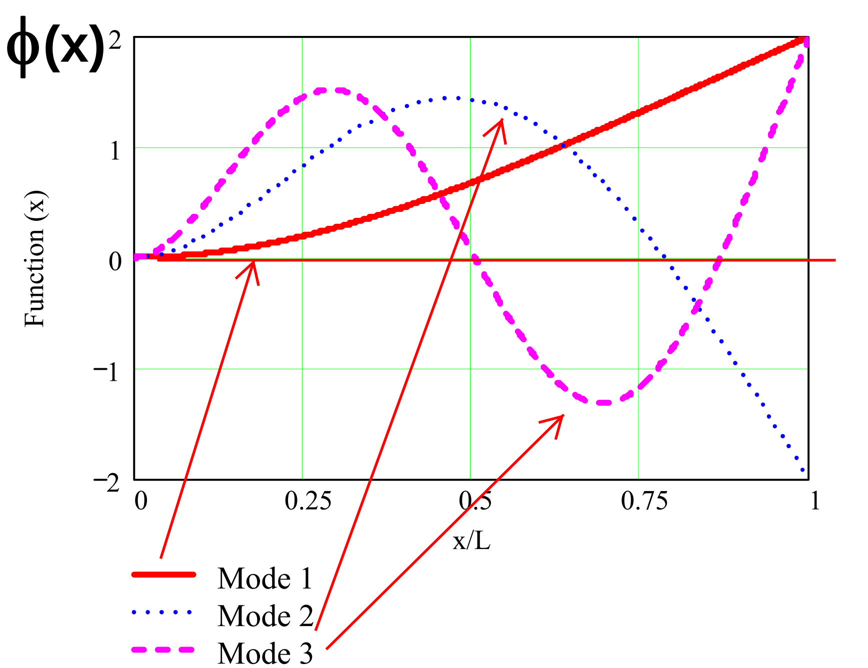
$$\phi_{(x)} = \cosh(\beta_i x) - \cos(\beta_i x) - \alpha_i\left[\sinh(\beta_i x) - \sin(\beta_i x)\right] \quad :$$
(62)

where
$$\alpha_i = \frac{\cosh(\beta_i L) + \cos(\beta_i L)}{\sinh(\beta_i L) + \sin(\beta_i L)} \tag{63}$$

and

$$\begin{aligned} \beta_1 L &= 1.875104 \rightarrow \alpha_1 = 0.734096 \\ \beta_2 L &= 4.694041 \rightarrow \alpha_2 = 1.018466 \\ \beta_3 L &= 7.854757 \rightarrow \alpha_3 = 0.999225 \\ &etc \end{aligned} \tag{64}$$

$$\phi(\beta, x) := \cosh(\beta \cdot x) - \cos(\beta \cdot x) - \alpha(\beta) \cdot (\sinh(\beta \cdot x) - \sin(\beta \cdot x))$$



$$\beta = \begin{pmatrix} 1.875 \\ 4.694 \\ 7.855 \end{pmatrix} \frac{1}{L}$$

$$\alpha\_ = \begin{pmatrix} 0.734 \\ 1.018 \\ 0.999 \end{pmatrix}$$

Fig. Natural mode shapes ϕ(x) for cantilever beam (fixed end-free end)

Properties of the natural modes

Recall that the pair $\left\{\lambda_k, \phi_{(x)_k}\right\}_{k=1,\ldots\infty}$ satisfy the ODE

$$\phi^{iv}{}_k - \lambda_k^2\, \phi_k = 0 \qquad k=1,2\ldots\infty \tag{65}$$

where

$$\beta_k^4 = \lambda_k^2 = \omega_k^2 \left(\frac{\rho A}{E I}\right)$$

As in the case of axial vibrations of a bar, it is easy[2] to show that the natural modes $\left\{\phi_k\right\}_{k=1,2\ldots}$ of a flexing beam satisfy the following **ORTHOGONAL** properties:

$$\int_0^L \left(E A \phi_i'' \phi_j''\right) dx = \begin{cases} \mathrm{K}_i & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases} \tag{66a}$$

$$\int_0^L \left(\rho A \phi_i \phi_j\right) dx = \begin{cases} \mathrm{M}_i & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases} \tag{66b}$$

For $i=j$, the i_{th} natural frequency follows from

$$\omega_i^2 = \frac{\mathrm{K}_i}{\mathrm{M}_i} = \frac{\int_0^L \left(E A \left(\phi_i''\right)^2\right) dx}{\int_0^L \left(\rho A \phi_i^2\right) dx} \tag{67}$$

Where $\mathrm{K}_i, \mathrm{M}_i$ are the i_{th} mode *equivalent* stiffness and mass coefficients.

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[2] Demonstration with integration by parts (twice).

Note that $\{\phi_k\}_{k=1,2...}$ is a **COMPLETE SET** of orthogonal functions

Now, consider the **initial conditions** for

$$v_{(x,t)} = \sum_k \phi(x)_k \, \mathrm{v}(t)_k = \sum_{k=1}^{\infty} \phi_{(x)_k} \left[C_k \cos(\omega_k t) + S_k \sin(\omega_k t) \right]$$

$$v_{(x,0)} = V_{(x)} = \sum_{k=1}^{\infty} \phi_k \, C_k; \quad \dot{v}_{(x,0)} = \dot{V}_{(x)} = \sum_{k=1}^{\infty} \phi_k \omega_k \, S_k \qquad (68)$$

Using the orthogonality properties, the coefficients (C_m, S_m) follow from

$$C_m = \frac{\int_0^L \left(\rho A \phi_m V_{(x)} \right) dx}{\mathrm{M}_m}, \quad m=1,2,...\infty \qquad (69a)$$

And similarly

$$S_m = \frac{\int_0^L \left(\rho A \phi_m \dot{V}_{(x)} \right) dx}{\omega_m \mathrm{M}_m}, \quad m=1,2,...\infty \qquad (69b)$$

This concludes the procedure to obtain the full solution for the lateral vibrations of a beam, i.e.

$$v_{(x,t)} = \sum_{k=1}^{\infty} \phi_{(x)_k} \left[C_k \cos(\omega_k t) + S_k \sin(\omega_k t) \right] \qquad (70)$$

Forced lateral vibrations of a beam

Consider a beam subjected to an arbitrary forcing function $f_{(x,t)}$. The PDE describing the lateral motions of the beam is

$$(\rho A)\frac{\partial^2 v}{\partial t^2} = f_{(x,t)} - \frac{\partial^2}{\partial x^2}\left(E I \frac{\partial^2 v}{\partial x^2}\right) \qquad (44)$$

Let $\{\phi_k\}_{k=1,2...}$ be the set of natural modes satisfying the boundary conditions of the beam configuration (pin-pin, fixed-free ends, etc). A solution to Eq. (44) is of the form

$$v_{(x,t)} = \sum_{k=1}^{\infty} \phi_{(x)_k}\, q_{(t)_k} \qquad (71)$$

Since the set $\{\phi_k\}_{k=1,2...}$ is complete, then any arbitrary function $f_{(x,t)}$ can be written as

$$f_{(x,t)} = \sum_{k=1}^{\infty} \phi_{(x)_k}\, Q_{(t)_k} \qquad (72)$$

where

$$Q_m = \frac{\int_0^L \left(\rho A \phi_m\, f_{(x,t)}\right) dx}{\mathrm{M}_m}, \quad m=1,2,...\infty \qquad (73)$$

Substitution of Eqs. (71, 72) into Eq. (44) $(\rho A)\frac{\partial^2 v}{\partial t^2} = f_{(x,t)} - \frac{\partial^2}{\partial x^2}\left(E I \frac{\partial^2 v}{\partial x^2}\right)$

gives

$$\sum_{k=1}^{\infty}\left[\rho A \phi_k \ddot{q}_k - \phi_k Q_k + E I \phi_k^{iv} q_k\right] = 0 \qquad (74)$$

but recall that each of the normal modes satisfies $\phi_k^{iv} - \lambda_k^2 \phi_k = 0$; and hence, Eq. (74) can be written as

$$\sum_{k=1}^{\infty}\left[\rho A \ddot{q}_k - Q_k + E I \lambda_k^2 q_k\right]\phi_k = 0$$

and, since the natural modes are linearly independent, then it follows that

$$\rho A \ddot{q}_k - Q_k + E I \lambda_k^2 q_k = 0 \quad {}_{k=1,2,....\infty} \qquad (75)$$

Lastly, recall that $\lambda_k^2 = \omega^2 \left(\frac{\rho A}{E I}\right)$; then $\lambda_k^2 E I = \omega^2 \rho A$, and write (75) as

$$\ddot{q}_k + \omega_k^2 q_k = Q_k / \rho A \quad ; \quad {}_{k=1,2,....\infty} \qquad (76)$$

Which can be easily solved for all type of excitations $Q_{(t)_k}$

[See solution of undamped SDOF EOMS – Lectures #2]

Example 3. Free-free ends beam

Recall $\phi_{(x)} = C_1 \cos(\beta x) + C_2 \sin(\beta x) + C_3 \cosh(\beta x) + C_4 \sinh(\beta x)$

The BCs are:

At x=0

$$M = \partial^2 v / \partial x^2 = 0 = \phi''_{(0)} \mathrm{v}_{(t)} \Rightarrow \quad \phi''_{(0)} = 0$$

$$\rightarrow \qquad \phi''_{(0)} = -C_1 + C_3 \qquad \text{(a)}$$

$$S_{x=0} = \partial^3 v / \partial x^3 = 0 = \phi'''_{(0)} \mathrm{v}_{(t)} \Rightarrow \quad \phi'''_{(0)} = 0$$

$$\rightarrow \qquad \phi'''_{(0)} = -C_2 + C_4 \qquad \text{(b)}$$

At x=L

$$M_{x=L} = \partial^2 v / \partial x^2 = 0 = \phi''_{(L)} \mathrm{v}_{(t)} \Rightarrow \quad \phi''_{(L)} = 0 \qquad (61.c)$$

$\rightarrow$

$$\phi''_{(L)} = 0 = -C_1 \cos(\beta L) - C_2 \sin(\beta L) + C_3 \cosh(\beta L) + C_4 \sinh(\beta L)$$

(c)

$$S_{x=L} = \partial^3 v / \partial x^3 = 0 = \phi'''_{(L)} \mathrm{v}_{(t)} \Rightarrow \quad \phi'''_{(L)} = 0$$

$\rightarrow$

$$\phi'''_{(L)} = 0 = C_1 \sin(\beta L) - C_2 \cos(\beta L) + C_3 \sinh(\beta L) + C_4 \cosh(\beta L)$$

(d)

Solution of Eqs. (a)-(d) gives

$$\phi_{(x)} = \cosh(\beta_i x) + \cos(\beta_i x) - \alpha_i \left[\sinh(\beta_i x) + \sin(\beta_i x) \right]$$

correction

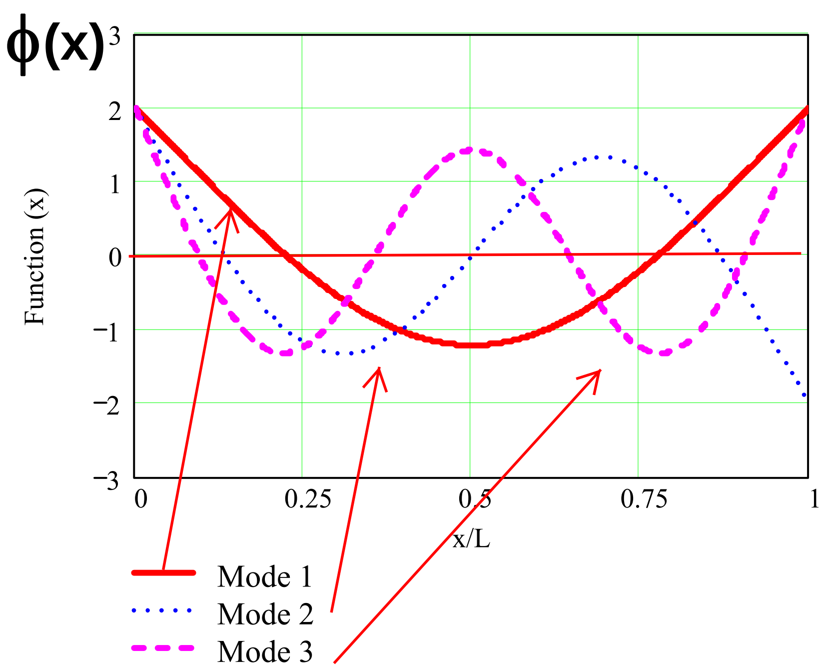
where
$$\alpha_i = \frac{\cosh(\beta_i L) - \cos(\beta_i L)}{\sinh(\beta_i L) - \sin(\beta_i L)} \qquad (63)$$

and

$$\beta_1 L = 4.730041 \rightarrow \alpha_1 = 0.982502$$
$$\beta_2 L = 7.853205 \rightarrow \alpha_2 = 1.000777 \qquad (64)$$
$$\beta_3 L = 10.99560 \rightarrow \alpha_3 = 0.999966$$
$$etc$$

Note that the lowest natural frequency is actually zero, i.e. a rigid body mode. β_0=0 & $\phi_0 = 1$

$$\phi(\beta, x) := \cosh(\beta \cdot x) + \cos(\beta \cdot x) - \alpha(\beta)\cdot(\sinh(\beta \cdot x) + \sin(\beta \cdot x))$$



$$\beta = \begin{pmatrix} 4.73 \\ 7.853 \\ 10.996 \end{pmatrix} \frac{1}{L}$$

$$\alpha\_ = \begin{pmatrix} 0.983 \\ 1.001 \\ 1 \end{pmatrix}$$

Fig. Elastic natural mode shapes ϕ(x) for beam with free-free ends

Characteristic (mode shape) equation for beams:

$$\phi_{(x)} = C_1 \cos(\beta x) + C_2 \sin(\beta x) + C_3 \cosh(\beta x) + C_4 \sinh(\beta x)$$

$$\phi' / \beta = -C_1 \sin(\beta x) + C_2 \cos(\beta x) + C_3 \sinh(\beta x) + C_4 \cosh(\beta x)$$

$$\phi'' / \beta^2 = -C_1 \cos(\beta x) - C_2 \sin(\beta x) + C_3 \cosh(\beta x) + C_4 \sinh(\beta x)$$

$$\phi''' / \beta^3 = C_1 \sin(\beta x) - C_2 \cos(\beta x) + C_3 \sinh(\beta x) + C_4 \cosh(\beta x)$$

Students:
The following pages contain five worked examples for prediction of the vibration response of bars, rods, strings, and beams.

Axial vibrations of elastic bar

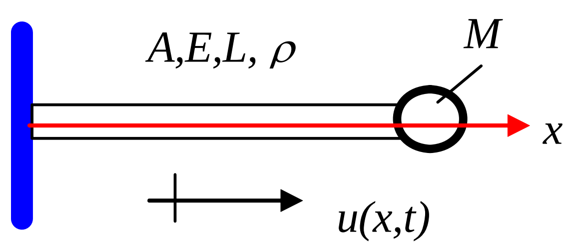
LSan Andres (c) SP 08 MEEN 617

ORIGIN := 1

The figure shows an elastic bar of length *L* and cross-sectional area *A*, and with density and elastic modulus equal to ρ and *E*, respectively. The bar is rigidly attached to a wall at its left end. At its right end, a rigid block or lumped mass *M* is firmly attached. Note that $M/M_{bar} = \varepsilon = 0.5$.
The field equation for axial motions *u(x,t)* of the bar is $\rho A \frac{\partial^2 u}{\partial t^2} = E A \frac{\partial^2 u}{\partial x^2}$

a) Determine the first three natural frequencies and characteristic modes (graph the modes) of the bar as a function of (ρ, *E*, *L*).
b) Using your experience, estimate the first natural frequency of the bar and block. Explain your assumptions. How good is the estimate when compared to the ones derived in (a)?

Solution Procedure



using separation of variables,

$$u(x,t) = \phi(x) \cdot v(t) \quad (1)$$

leads to the following two ODEs:

$$\frac{d^2}{dx^2}\phi + \lambda^2 \cdot \phi = 0 \quad (2a)$$

and

$$\frac{d^2}{dt^2} v + \Omega^2 \cdot v = 0 \quad (2b)$$

where $\lambda = \Omega \cdot \left(\frac{\rho}{E}\right)^{.5}$

The solution to the ODEs is simple, i.e.:

$$v(t) = A_t \cdot \cos(\Omega \cdot t) + B_t \cdot \sin(\Omega \cdot t)$$

$$\phi(x) = A_x \cdot \cos(\lambda \cdot x) + B_x \cdot \sin(\lambda \cdot x) \quad (3)$$

Satisfy the boundary conditions. At x=0, u(0,t)=0 (**fixed end**). Thus

$$\phi(0) = A_x \cdot \cos(\lambda \cdot 0) + B_x \cdot \sin(\lambda \cdot 0)$$

$$\phi(0) = A_x \cdot 1 + B_x \cdot 0$$

Then: $A_x = 0$

and

$$\phi(x) = \sin(\lambda \cdot x) \quad (4)$$

At the right end, x=L, the appropriate boundary condition is: axial force = M accel

$$-E \cdot A \cdot \frac{d}{dx} u = M \cdot \frac{d^2}{dt^2} u \quad (5a)$$

or: $-E \cdot A \cdot v \cdot \frac{d}{dx}\phi = M \cdot \phi \cdot \frac{d^2}{dt^2} v$. Noting that $\frac{d^2}{dt^2} v = -\Omega^2 \cdot v$ from (2a)

then, at **x=L**

$$-E \cdot A \cdot \frac{d}{dx}\phi = -M \cdot \Omega^2 \cdot \phi(L) \quad \text{(5b)}$$

recall $\lambda^2 = \Omega^2 \cdot \left(\frac{\rho}{E}\right)$ ---> $E \cdot A \cdot \frac{d}{dx}\phi = M \cdot \lambda^2 \cdot \frac{E}{\rho} \cdot \phi(L)$ (5b)

Replacing (4) $\phi(x) = \sin(\lambda \cdot x)$ & $\frac{d}{dx}\phi = \lambda \cdot \cos(\lambda \cdot x)$ into (5b) gives

$$\frac{\rho \cdot A \cdot L}{M} \cdot \cos(\bar{\lambda}) - \bar{\lambda} \cdot \sin(\bar{\lambda}) = 0 \quad \text{where} \quad \bar{\lambda} = \lambda \cdot L$$

define

$\varepsilon = \frac{M}{(\rho \cdot A \cdot L)}$, and write the **characteristic equation** as: $\tan(\bar{\lambda}) = \frac{1}{\varepsilon \cdot \bar{\lambda}}$ (6)

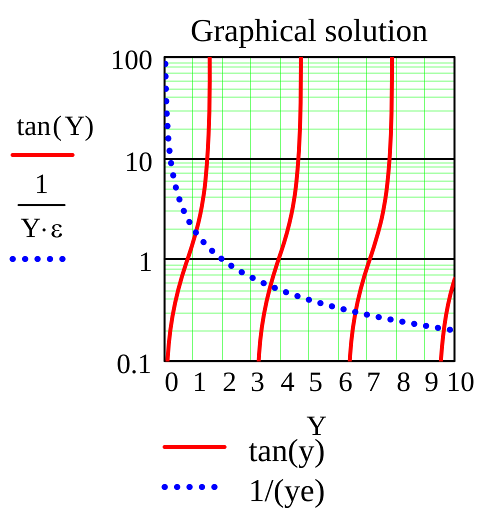
from experience or having worked other problems, using a calculator, $\varepsilon := 0.5$

guess values $y := \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ $n := 3$ # of roots $f(y) := \tan(y) - \frac{1}{y \cdot \varepsilon}$

$\lambda\_ := \overrightarrow{root(f(y), y)}$ $\lambda\_ = \begin{pmatrix} 1.077 \\ 3.643 \\ 6.578 \end{pmatrix}$ where $\lambda\_ = \lambda \cdot L$ & $\Omega = \lambda \cdot \left(\frac{E}{\rho}\right)^{.5}$

And thus, the first three natural frequencies are

$$\Omega = \begin{pmatrix} 1.077 \\ 3.643 \\ 6.578 \end{pmatrix} \cdot \frac{1}{L} \cdot \left(\frac{E}{\rho}\right)^{.5}$$

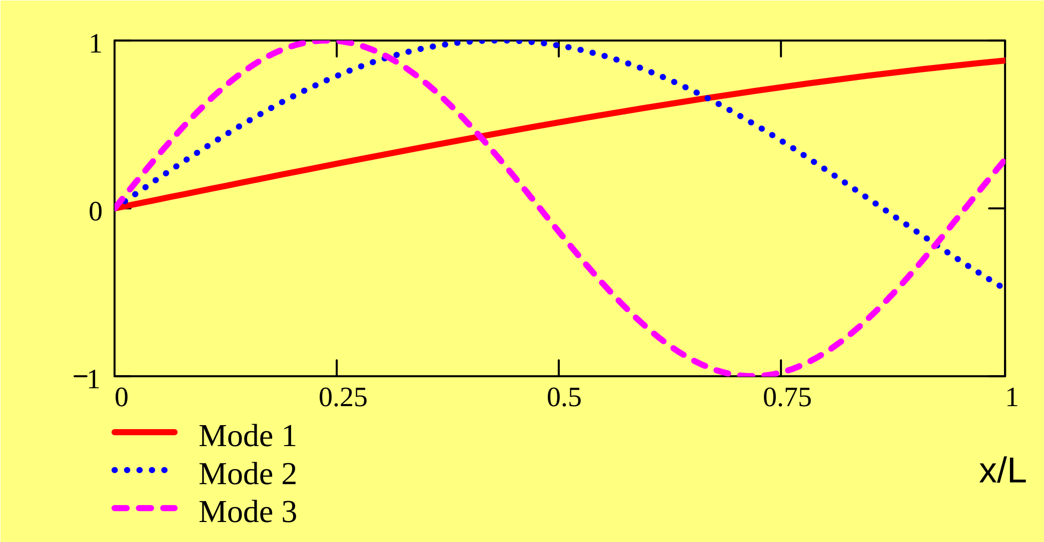


The shape functions are

$\phi1(z) := \sin(\lambda\_{1} \cdot z)$

$\phi2(z) := \sin(\lambda\_{2} \cdot z)$ where $z = \frac{x}{L}$

$\phi3(z) := \sin(\lambda\_{3} \cdot z)$



(b) Approximate first natural frequency. Using mode shape $\varphi(x) = \frac{x}{L}$

recal lots of problems worked in class and homeworks, one can easily estimate the equivalent stiffness and mass as

$$K_{eq} = \frac{A \cdot E}{L} \qquad Meq = \left(\frac{\rho \cdot A \cdot L}{3} + M\right) = \rho \cdot A \cdot L \cdot \left(\frac{1}{3} + \varepsilon\right)$$

and the estimation for natural frequency is

$$\omega_{1approx} = \left(\frac{K_{eq}}{M_{eq}}\right)^{.5} = \frac{1}{L} \cdot \left(\frac{E}{\rho}\right)^{.5} \cdot \left(\frac{3}{1 + \varepsilon \cdot 3}\right)^{.5}$$

$$\left(\frac{3}{1 + \varepsilon \cdot 3}\right)^{.5} = 1.095$$

$$\omega_{1approx} = 1.095 \cdot \left[\frac{1}{L} \cdot \left(\frac{E}{\rho}\right)^{.5}\right]$$

which is just 2.5% higher than the exact value

$$y_1 = 1 \qquad \left[\frac{1}{L} \cdot \left(\frac{E}{\rho}\right)^{.5}\right]$$

(c) Free vibrations response:

Consider the following intitial conditions,

at t=0 $u(x,0) = a \cdot \frac{x}{L}$ $a := 0.01$ **Uniform axial stretching**

$\frac{d}{dt}u = 0$ no velocity = rest condition

The response for axial motions of the bar is

$$u(x,t) = \sum_{i=1}^{\blacksquare} \phi_i \cdot \left(A_i \cdot \cos(\Omega_i \cdot t) + B_i \cdot \sin(\Omega_i \cdot t)\right)$$

Since the initial velocity = 0 everywhere, then it follows that $B_i = 0$

and $u(x,0) = a \cdot \frac{x}{L} = \sum_{i=1}^{\blacksquare} \phi_i \cdot (A_i)$

multiplying this equation by ϕj and integrating over the domain gives

$$A_i = \frac{\int_0^L \left(a \cdot \frac{x}{L}\right) \cdot \phi_i \, dx}{\int_0^L \phi_i^2 \, dx}$$

"Used orthogonality property of shape functions

Define $z = \frac{x}{L}$ $\phi_i(z) := \sin(\lambda\_{i} \cdot z)$

recall:

$\lambda\_ = \lambda \cdot L$

$$\lambda\_ = \begin{pmatrix} 1.077 \\ 3.643 \\ 6.578 \end{pmatrix}$$

$i := 1..n$

$$A_i := \frac{\left(\int_0^1 z \cdot \sin(\lambda\_{i} \cdot z) \, dz\right) \cdot a}{\int_0^1 \sin(\lambda\_{i} \cdot z)^2 \, dz}$$

$$A = \begin{pmatrix} 0.01 \\ 4.625 \times 10^{-3} \\ -2.897 \times 10^{-3} \end{pmatrix}$$

$$u(x,t) := \sum_{i=1}^{n} \sin\left(\lambda\_{i} \cdot \frac{x}{L}\right) \cdot \left(A_i \cdot \cos(\Omega_i \cdot t)\right)$$

Calculate time response at various spatial points in bar:

PHYSICAL Parameters: $E := 20 \cdot 10^9 \cdot \frac{N}{m^2}$ $\rho := 7800 \cdot \frac{kg}{m^3}$ $L := 1 \cdot m$ $d := 0.1 \cdot m$ $\varepsilon = 0.5$

$$\Omega := \lambda\_ \cdot \frac{1}{L} \cdot \left(\frac{E}{\rho}\right)^{.5}$$

$$f_n := \frac{\Omega}{2 \cdot \pi} \qquad f_n = \begin{pmatrix} 274.443 \\ 928.439 \\ 1.676 \times 10^3 \end{pmatrix} Hz$$

natural frequencies

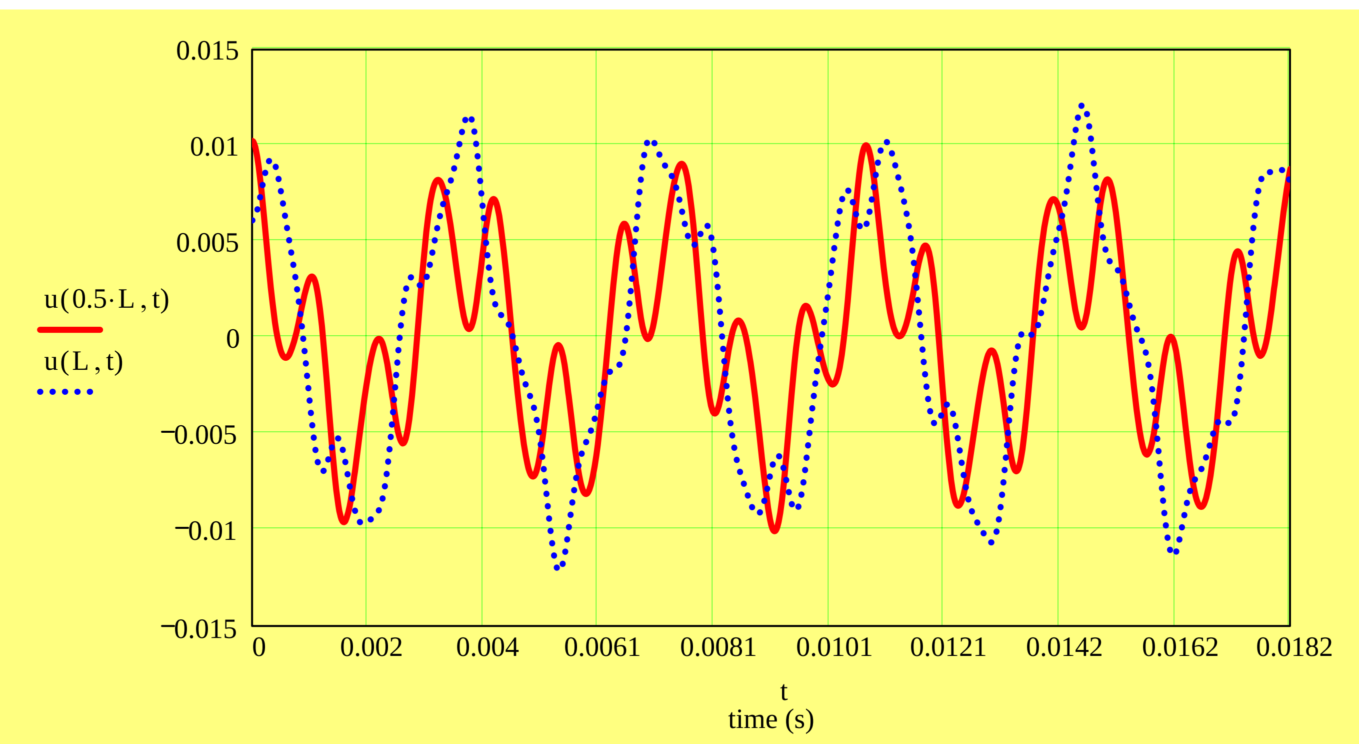
$$T_n := \frac{1}{f_n} \qquad T_n^T = \left(3.644 \times 10^{-3} \quad 1.077 \times 10^{-3} \quad 5.965 \times 10^{-4}\right) s$$

natural periods

$$u(x,t) := \sum_{i=1}^{n} \sin\left(\lambda\_i \cdot \frac{x}{L}\right) \cdot \left(A_i \cdot \cos\left(\Omega_i \cdot t\right)\right)$$

Response at bar: midpoint & end

$$T_{max} := 5 \cdot T_{n_1}$$



Axial vibrations of elastic bar (2)

The figure shows an elastic bar of length *L* and cross-sectional area *A*, and with density and elastic modulus equal to ρ and *E*, respectively. The bar is rigidly attached to a wall at its left end. At its right end, a massless spring K_s connects the bar to another fixed wall. $K_S L/(EA) = \varepsilon = 0.25$.

The field equation for axial motions *u(x,t)* of the bar is $\rho A \frac{\partial^2 u}{\partial t^2} = E A \frac{\partial^2 u}{\partial x^2}$

a) Determine the first TWO natural frequencies and characteristic modes (graph the modes) of the bar as a function of (ρ, *E*, *L*, and ε). [20]

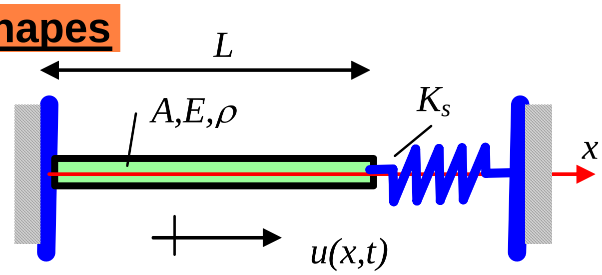
b) Using your experience, <u>estimate</u> the first natural frequency of the bar and block. Explain your assumptions. How good is the estimate when compared to the ones derived in (a)? [5]

ORIGIN := 1

PHYSICAL Parameters: $E := 20 \cdot 10^9 \cdot \frac{N}{m^2}$ $\rho := 7800 \cdot \frac{kg}{m^3}$ $L := 1 \cdot m$ $d := 0.1 \cdot m$

$$A := \frac{\pi \cdot d^2}{4} \qquad \frac{A \cdot E}{L} = 1.571 \times 10^8 \frac{N}{m}$$

(a) natural frequencies & mode shapes



using separation of variables,

$$u(x,t) = \phi(x) \cdot v(t) \qquad (1)$$

leads to the following two ODEs:

$$\frac{d^2}{dx^2}\phi + \lambda^2 \cdot \phi = 0 \qquad (2a)$$

and

$$\frac{d^2}{dt^2}v + \Omega^2 \cdot v = 0 \qquad (2b)$$

where $\lambda = \Omega \cdot \left(\frac{\rho}{E}\right)^{.5}$

The solution to the ODEs is simple, i.e.:

$$v(t) = A_t \cdot \cos(\Omega \cdot t) + B_t \cdot \sin(\Omega \cdot t)$$

(3)

$$\phi(x) = A_x \cdot \cos(\lambda \cdot x) + B_x \cdot \sin(\lambda \cdot x)$$

Satisfy the boundary conditions. At x=0, u(0,t)=0 (**fixed end**). Thus

$$\phi(0) = A_x \cdot \cos(\lambda \cdot 0) + B_x \cdot \sin(\lambda \cdot 0)$$

$$\phi(0) = A_x \cdot 1 + B_x \cdot 0$$

Then: $A_x = 0$

and

$$\phi(x) = \sin(\lambda \cdot x) \qquad (4)$$

At the right end, x=L, the appropriate boundary condition is: **bar axial force = Ks u = spring force**

at x=L

$$-E\cdot A\cdot\frac{d}{dx}u = k_S\cdot u \qquad (5a)$$

or: $-E\cdot A\cdot v\cdot\frac{d}{dx}\phi = k_S\cdot\phi\cdot v$

$$-E\cdot A\cdot\frac{d}{dx}\phi = k_S\cdot\phi(L) \qquad (5b) \quad \text{from (2a)}$$

Replacing (4) $\phi(x) = \sin(\lambda\cdot x)$ & $\frac{d}{dx}\phi = \lambda\cdot\cos(\lambda\cdot x)$ into (5b) gives

$$\frac{E\cdot A}{k_S\cdot L}\cdot\bar{\lambda}\cdot\cos(\bar{\lambda}) + \sin(\bar{\lambda}) = 0$$

where $\bar{\lambda} = \lambda\cdot L$

define

$$\varepsilon = \frac{k_S}{\frac{E\cdot A}{L}}$$

, and write the **characteristic equation** as: $\tan(\bar{\lambda}) + \frac{1}{\varepsilon}\cdot\bar{\lambda} = 0$ (6)

$\varepsilon := .25$

$n := 4$

# of roots

guess values from graphical soln $y := (1.71 \;\; 4.76 \;\; 7.88 \;\; 11)^T$

$f(y) := \tan(y) + \frac{1}{\varepsilon}\cdot y$

$\lambda\_ := \overrightarrow{\text{root}(f(y), y)}$

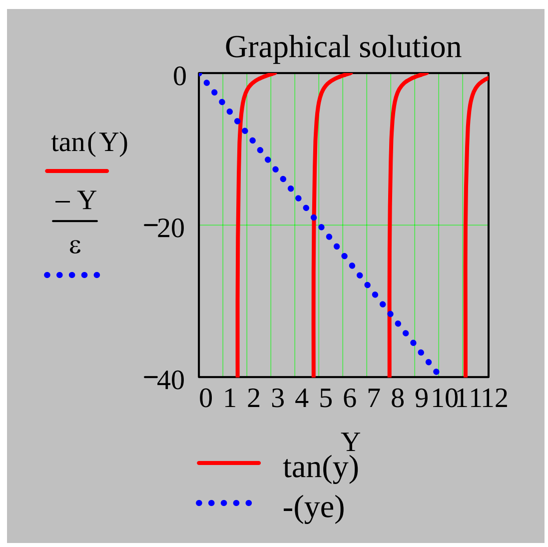
$$\lambda\_ = \begin{pmatrix} 1.716 \\ 4.765 \\ 7.886 \\ 11.018 \end{pmatrix}$$

where $\lambda\_ = \lambda\cdot L$ & $\Omega = \lambda\cdot\left(\frac{E}{\rho}\right)^{.5}$

And thus, the **first four natural frequencies** are

$$\Omega := \lambda\_\cdot\frac{1}{L}\cdot\left(\frac{E}{\rho}\right)^{.5}$$

$$\Omega^T = \left(2.747\times10^3 \;\; 7.63\times10^3 \;\; 1.263\times10^4 \;\; 1.764\times10^4\right)\frac{rad}{s}$$



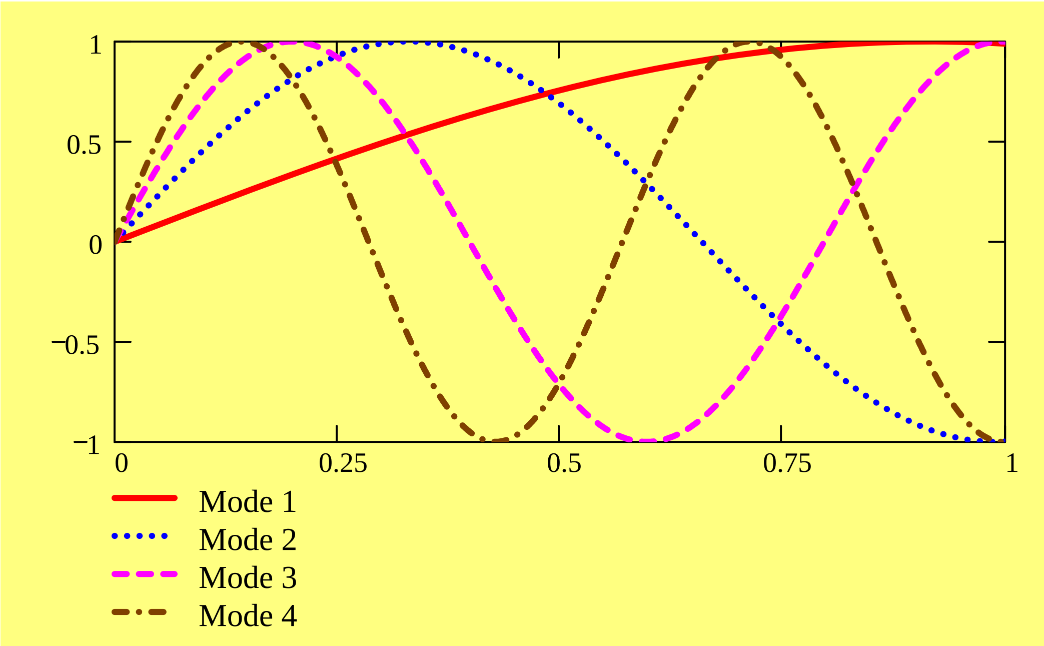
The shape functions are

$\phi1(z) := \sin(\lambda\_{1}\cdot z)$

$\phi2(z) := \sin(\lambda\_{2}\cdot z)$ where $z = \frac{x}{L}$

$\phi3(z) := \sin(\lambda\_{3}\cdot z)$

$\phi4(z) := \sin(\lambda\_{4}\cdot z)$



(b) Approximate first natural frequency. Using mode shape $\varphi(x) := \frac{x}{L}$ easily estimate the equivalent stiffness and mass as

$$K_{eq} := \int_0^L E\cdot A\cdot\left(\frac{d}{dx}\varphi(x)\right)^2 dx + k_S\cdot(\varphi(L))^2 \qquad M_{eq} := \left[\int_0^L \rho\cdot A\cdot(\varphi(x))^2\,dx\right]$$

$$K_{eq} = \frac{A\cdot E}{L} + k_S \qquad Meq = \left(\frac{\rho\cdot A\cdot L}{3}\right)$$

$$\varepsilon = \frac{k_S}{\frac{E\cdot A}{L}} \qquad K_{eq} = \frac{A\cdot E}{L}\cdot(1+\varepsilon)$$

$\varepsilon = 0.25$

and the estimation for the **fundamental natural frequency** is

$$\omega_{1approx} = \left(\frac{K_{eq}}{M_{eq}}\right)^{.5} = \frac{1}{L}\cdot\left(\frac{E}{\rho}\right)^{.5}\cdot\left(\frac{1+\varepsilon}{\frac{1}{3}}\right)^{.5}$$

$$a := \left(\frac{1+\varepsilon}{\frac{1}{3}}\right)^{.5} \qquad a = 1.936$$

which is just 13% higher than the exact value

$$\frac{a}{\lambda\_1} = 1.129 \qquad \lambda\_1 = 1.716\left[\frac{1}{L}\cdot\left(\frac{E}{\rho}\right)^{.5}\right]$$

(c) Free vibrations response:

Consider the following intitial conditions,

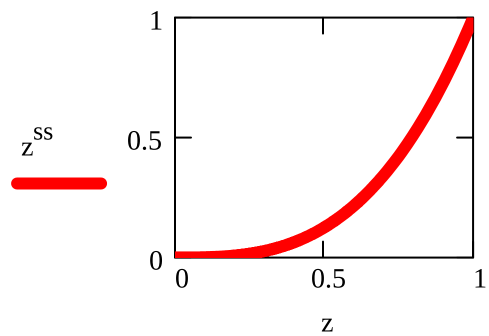
at t=0

$u(x,0) = 0$ **no axial stretching**

$$\frac{d}{dt}u = v_o \cdot \left(\frac{x}{L}\right)^{ss}$$ **velocity**

$$v_o := 1 \cdot \frac{m}{s}$$

$ss := 3$



The response for axial motions of the bar is

$$u(x,t) = \sum_{i=1}^{inf} \phi_i \cdot \left(A_i \cdot \cos(\Omega_i \cdot t) + B_i \cdot \sin(\Omega_i \cdot t)\right)$$

Since the initial displacement = 0 everywhere, then it follows

$A_i = 0$ and $$\frac{d}{dt}u(x,0) = v_o \cdot \left(\frac{x}{L}\right)^{s} = \sum_{i=1}^{inf} \phi_i \cdot (B_i \cdot \Omega_i)$$

multiplying this equation by ϕj and integrating over the domain gives

$$B_i \cdot \Omega_i = \frac{\int_0^L \left[v_o \cdot \left(\frac{x}{L}\right)^{ss}\right] \cdot \phi_i \, dx}{\int_0^L \phi_i^2 \, dx}$$ "Used orthogonality property of shape functions

Define

$$z = \frac{x}{L}$$

$\phi_i(z) := \sin(\lambda\_{i} \cdot z)$

$i := 1 .. n$

recall:

$\lambda\_ = \lambda \cdot L$

$$\lambda\_ = \begin{pmatrix} 1.716 \\ 4.765 \\ 7.886 \\ 11.018 \end{pmatrix}$$

$$B_i := \frac{v_o}{\Omega_i} \cdot \frac{\left(\int_0^1 z^{ss} \cdot \sin(\lambda\_{i} \cdot z)\, dz\right)}{\int_0^1 \sin(\lambda\_{i} \cdot z)^2 \, dz}$$

$$B^T = \left(1.586 \times 10^{-4} \quad -3.33 \times 10^{-5} \quad 7.935 \times 10^{-6} \quad -2.97 \times 10^{-6}\right) m$$

$$u(x,t) := \sum_{i=1}^{n} \sin\left(\lambda\_{i} \cdot \frac{x}{L}\right) \cdot \left[B_i \cdot \left(\sin(\Omega_i \cdot t)\right)\right]$$

Calculate time response at various spatial points in bar:

natural frequencies:

$$f_n := \frac{\Omega}{2\cdot\pi}$$

$$f_n^T = \left(437.199 \;\; 1.214\times10^3 \;\; 2.01\times10^3 \;\; 2.808\times10^3\right) \text{Hz}$$

natural periods

$$T_n := \frac{1}{f_n}$$

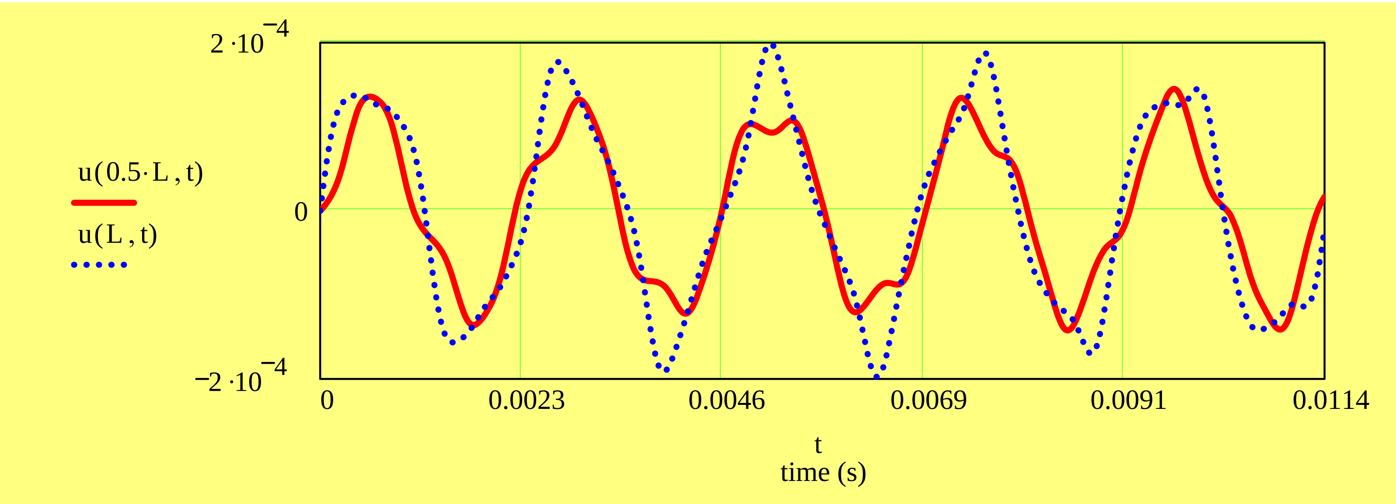
$$T_n^T = \left(2.287\times10^{-3} \;\; 8.235\times10^{-4} \;\; 4.976\times10^{-4} \;\; 3.561\times10^{-4}\right) \text{s}$$

Bar displacement: midpoint & end

$$u(x,t) := \sum_{i=1}^{n} \sin\left(\lambda\_i\cdot\frac{x}{L}\right)\cdot\left(B_i\cdot\sin\left(\Omega_i\cdot t\right)\right)$$

for graph only

$$T_{max} := 5\cdot T_{n_1}$$

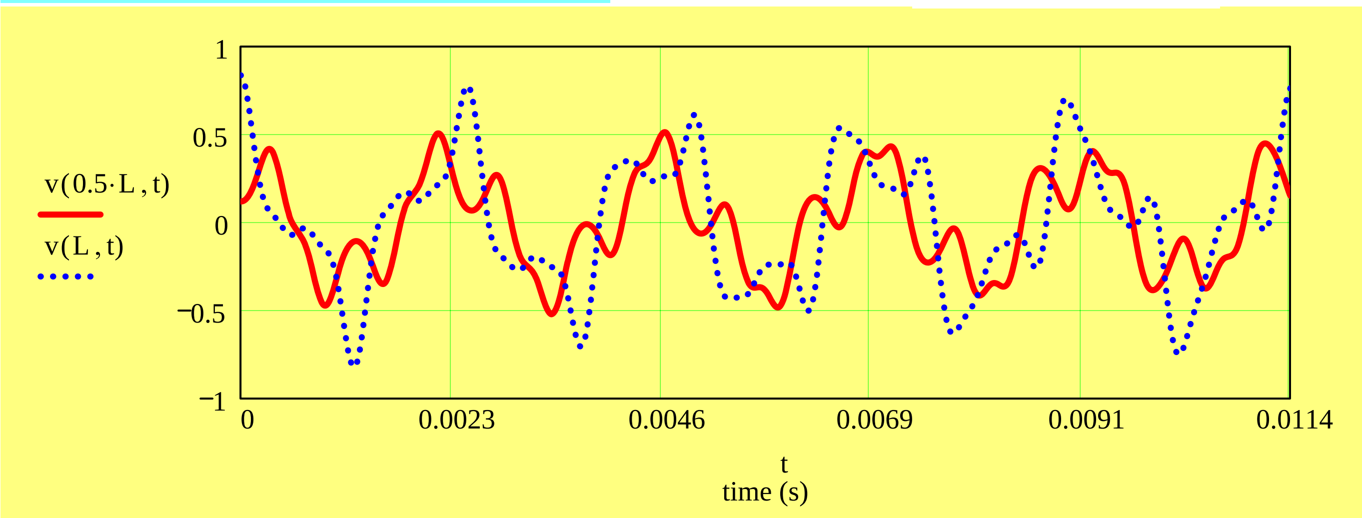


Bar velocity: midpoint & end

$$v(x,t) := \sum_{i=1}^{n} \sin\left(\lambda\_i\cdot\frac{x}{L}\right)\cdot\left(B_i\cdot\Omega_i\cdot\cos\left(\Omega_i\cdot t\right)\right)$$

$$v_o = 1\frac{m}{s} \qquad v_o\cdot\left(\frac{1}{2}\right)^{ss} = 0.125\frac{m}{s}$$

$$time := 0\cdot s \qquad v(L, time) = 0.837\frac{m}{s}$$



TORSIONAL VIBRATIONS OF AN ELASTIC ROD

The figure shows an elastic rod of length L , radius R, density ρ and elastic shear modulus G. The rod is rigidly attached to a wall at its left end. At the rod right end, a massless spring K_S connects the rod to a fixed wall. The field equation for angular motions $\Theta(x,t)$ of an elastic rod under torsion is

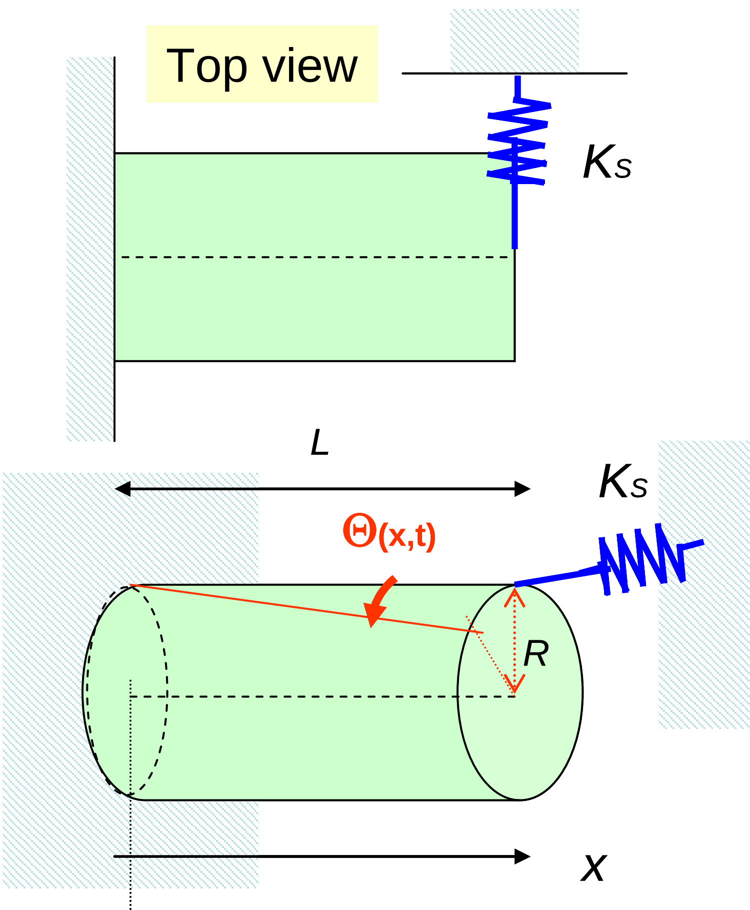
$\rho J \frac{\partial^2 \Theta}{\partial t^2} = G J \frac{\partial^2 \Theta}{\partial x^2}$ where $J = \frac{1}{2} \pi R^4$ is the polar moment of area and

$M = G J \frac{\partial \Theta}{\partial x}$ is the torsional moment. Let $K_S R^2 L/(GJ) = \boldsymbol{\varepsilon} = \mathbf{0.25}$.

a) Find the first TWO natural frequencies and characteristic modes (sketch the modes) of the bar as a function of (ρ, G, L, J and ε). [35]

b) Using experience, estimate the first natural frequency of the bar and spring. Explain your assumptions. How good is the estimate when compared to the exact (first) value derived in (a)? [15]

If needed use the following G=12 10^9 Pa, ρ=7800 kg/m^3, L=1m, R=0.05m



Torsional vibrations of elastic rod

The figure shows an elastic rod of length L , radius R, density ρ and elastic shear modulus G. The rod is rigidly attached to a wall at its left end. At the rod right end, a massless spring K_S connects the rod to a fixed wall. The field equation for angular motions $\Theta(x,t)$ of an elastic rod under torsion is

$\rho J \frac{\partial^2 \Theta}{\partial t^2} = G J \frac{\partial^2 \Theta}{\partial x^2}$ where $J = ½ \pi R^4$ is the polar moment of area and $M = G J \frac{\partial \Theta}{\partial x}$ is the torsional moment. Let $K_S R^2 L/(GJ) = \boldsymbol{\varepsilon = 0.25}$.

a) Find the first TWO natural frequencies and characteristic modes (sketch the modes) of the bar as a function of (ρ, G, L, J and ε). [35]

b) Using experience, <u>estimate</u> the first natural frequency of the bar and spring. <u>Explain your assumptions</u>. How good is the estimate when compared to the exact (first) value derived in (a)? [15]

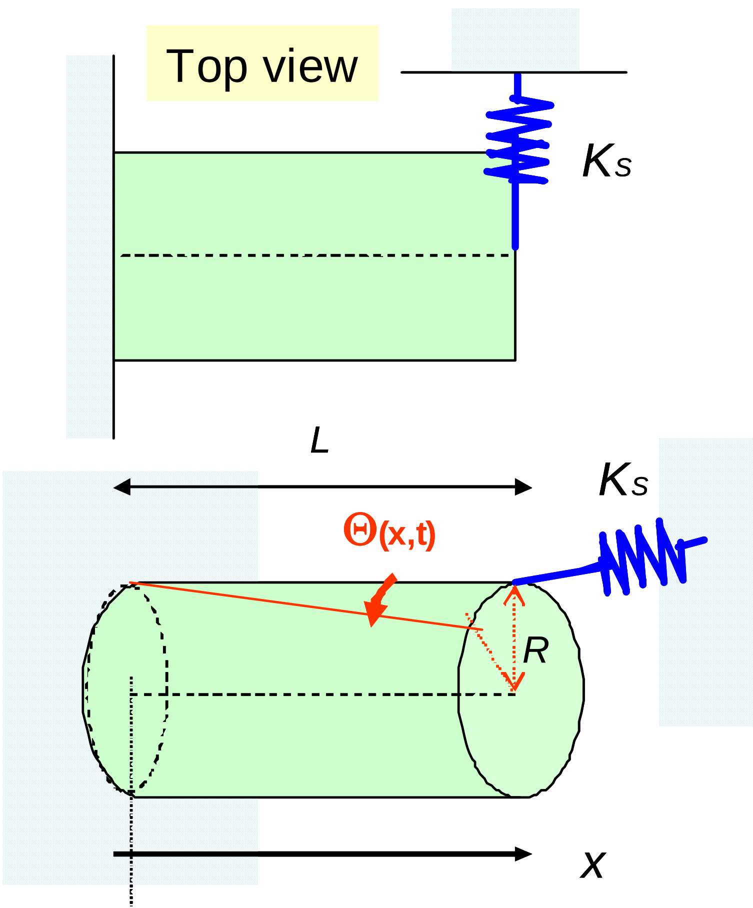
If needed use the following G=12 10^9 Pa, ρ=7800 kg/m^3, L=1m, R=0.05m

ORIGIN := 1

PHYSICAL Parameters:

$G := 12 \cdot 10^9 \cdot \frac{N}{m^2}$ $\quad \rho := 7800 \cdot \frac{kg}{m^3}$ $\quad L := 1 \cdot m$ $\quad d := 0.1 \cdot m$

$A := \frac{\pi \cdot d^2}{4}$ $\quad J = \int_0^r r^2 \, dA$ $\quad J := \frac{\pi \cdot d^4}{32}$ $\quad J = 9.817 \times 10^{-6} m^4$



$$\rho J \frac{\partial^2 \Theta}{\partial t^2} = G J \frac{\partial^2 \Theta}{\partial x^2} \qquad (0)$$

(a) natural frequencies & mode shapes

using separation of variables, $\Theta(x,t) = \phi(x)\cdot v(t)$ (1)

Substitute into the field Eq. (0) to obtain the following two ODEs:

$$\frac{d^2}{dx^2}\phi + \lambda^2\cdot\phi = 0 \quad (2a)$$

where $\lambda = \Omega\cdot\left(\frac{\rho}{G}\right)^{.5}$

and

$$\frac{d^2}{dt^2}v + \Omega^2\cdot v = 0 \quad (2b)$$

The solution to the ODEs is simple, i.e.:

$$v(t) = A_t\cdot\cos(\Omega\cdot t) + B_t\cdot\sin(\Omega\cdot t)$$

(3)

$$\phi(x) = A_x\cdot\cos(\lambda\cdot x) + B_x\cdot\sin(\lambda\cdot x)$$

Satisfy the boundary conditions. At x=0, Θ(0,t)=0 (**fixed end - no angular deformation**). Thus

$$\phi(0) = A_x\cdot\cos(\lambda\cdot 0) + B_x\cdot\sin(\lambda\cdot 0)$$

$$\phi(0) = A_x\cdot 1 + B_x\cdot 0$$

Then: $A_x = 0$

and $\phi(x) = \sin(\lambda\cdot x)$ (4)

At the right end, x=L, the appropriate boundary condition is: **rod torsional moment = spring reaction force x moment arm**

at x=L

$$-G\cdot J\cdot\frac{d}{dx}\Theta = (k_S\cdot u_L)\cdot R \quad (5a)$$

$u_L = \Theta_L\cdot R$ $\quad R := \frac{d}{2}$

or: $-G\cdot J\cdot v\cdot\frac{d}{dx}\phi = k_S\cdot R^2\cdot\phi\cdot v$

$$-G\cdot J\cdot\frac{d}{dx}\phi = k_S\cdot R^2\cdot\phi(L) \quad (5b)$$

from (2a)

Replacing (4) $\phi(x) = \sin(\lambda\cdot x)$ & $\frac{d}{dx}\phi = \lambda\cdot\cos(\lambda\cdot x)$ into (5b) gives

$$\frac{G\cdot J}{k_S\cdot R^2\cdot L}\cdot\bar{\lambda}\cdot\cos(\bar{\lambda}) + \sin(\bar{\lambda}) = 0$$

where $\bar{\lambda} = \lambda\cdot L$

define

$$\varepsilon = \frac{k_S\cdot R^2\cdot L}{G\cdot J}$$

, and write the **characteristic equation** as: $\tan(\bar{\lambda}) + \frac{1}{\varepsilon}\cdot\bar{\lambda} = 0$ (6)

$\varepsilon := 0.25$

$n := 4$

\# of roots

guess values from graphical soln $y := (1.71 \;\; 4.76 \;\; 7.88 \;\; 11)^T$

$$f(y) := \tan(y) + \frac{1}{\varepsilon}\cdot y$$

$$\lambda\_ := \overrightarrow{root(f(y), y)}$$

$$\lambda\_ = \begin{pmatrix} 1.716 \\ 4.765 \\ 7.886 \\ 11.018 \end{pmatrix}$$

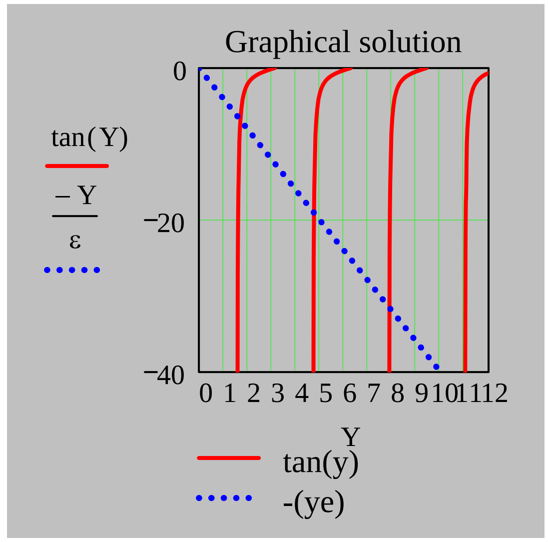
where

$\lambda\_ = \lambda \cdot L$ & $\Omega = \lambda \cdot \left(\frac{E}{\rho}\right)^{.5}$

And thus, the **first four natural frequencies** are

$$\Omega := \lambda\_ \cdot \frac{1}{L} \cdot \left(\frac{G}{\rho}\right)^{.5}$$

$$\Omega^T = \left(2.128 \times 10^3 \quad 5.91 \times 10^3 \quad 9.781 \times 10^3 \quad 1.367 \times 10^4\right) \frac{rad}{s}$$



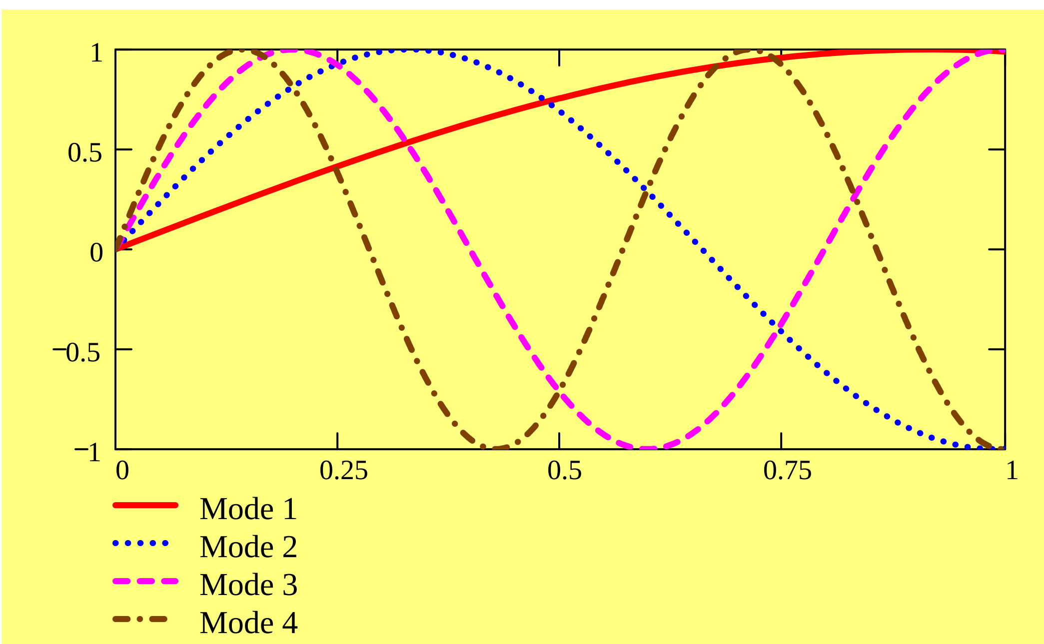
The shape functions are

$$\phi1(z) := \sin(\lambda\_{1} \cdot z)$$

$$\phi2(z) := \sin(\lambda\_{2} \cdot z) \quad \text{where} \quad z = \frac{x}{L}$$

$$\phi3(z) := \sin(\lambda\_{3} \cdot z)$$

$$\phi4(z) := \sin(\lambda\_{4} \cdot z)$$



(b) Approximate first natural frequency. Using mode shape $\varphi(x) := \frac{x}{L}$

easily estimate the equivalent torsional stiffness and mass moment of inertia from

$$K_{\Theta eq} := \int_0^L G\cdot J\cdot\left(\frac{d}{dx}\varphi(x)\right)^2 dx + k_S\cdot(R\cdot\varphi(L))^2 \qquad I_{eq} := \left[\int_0^L \rho\cdot J\cdot(\varphi(x))^2\,dx\right]$$

$$K_{\Theta eq} = \frac{G\cdot J}{L} + k_S\cdot R^2 \qquad K_{\Theta eq} = \frac{G\cdot J}{L}\cdot(1+\varepsilon) \qquad Ieq = \left(\frac{\rho\cdot J\cdot L}{3}\right)$$

with $\varepsilon = \frac{k_S\cdot R^2\cdot L}{G\cdot J}$ $\qquad \varepsilon = 0.25$

and the estimation for the **fundamental natural frequency** is

$$\omega_{1approx} = \left(\frac{K_{\Theta eq}}{I_{eq}}\right)^{.5} = \frac{1}{L}\cdot\left(\frac{G}{\rho}\right)^{.5}\cdot a \qquad \text{where} \qquad a := \left(\frac{1+\varepsilon}{\frac{1}{3}}\right)^{.5} \qquad a = 1.936$$

compare to the exact value: $\lambda\_1 = 1.716$ $\left[\frac{1}{L}\cdot\left(\frac{G}{\rho}\right)^{.5}\right]$

$\frac{a}{\lambda\_1} = 1.129$, i.e just 13 % higher than the exact value

$$k_S := \frac{\varepsilon\cdot G\cdot J}{R^2\cdot L}$$

$$k_S = 1.178\times 10^7\,\frac{N}{m}$$

Note: do realize the torsional bar vibration problem is dimensionally and physically equivalent to the axial vibrations of an elastic bar

lateral vibrations of elastic beam

LSan Andres (c) SP 08 MEEN 617

$d := 15 \cdot mm$ diameter

$L := 1 \cdot m$ length

$\rho := 7800 \cdot \frac{kg}{m^3}$

$E := 2 \cdot 10^{11} \cdot \frac{N}{m^2}$

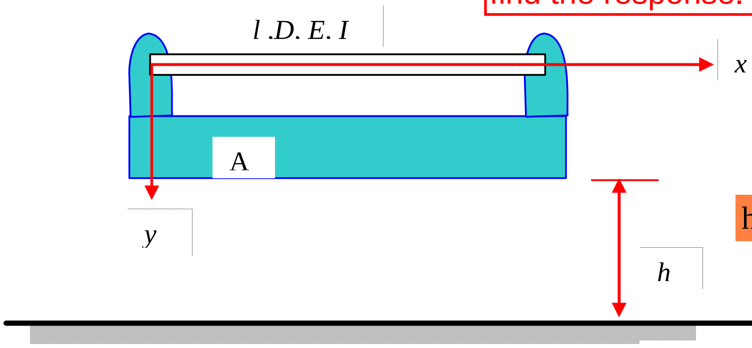
$ORIGIN := 1$

beam and frame are dropped from height h. find the response.

$A := \frac{\pi \cdot d^2}{4}$ cross-sectional area

$I_P := \frac{\pi \cdot d^4}{64}$

area moment of inertia (polar)



$h := 1 \cdot m$

Solution Procedure

free fall velocity $v_o := (2 \cdot g \cdot h)^{.5}$

using separation of variables,

$u(x,t) = \phi(x) \cdot v(t)$ (1)

$$(\rho A)\frac{\partial^2 v}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left(E\,I\,\frac{\partial^2 v}{\partial x^2}\right)$$

leads to the following two ODEs:

$$\frac{d^4}{dx^4}\phi - \lambda^2 \cdot \phi = 0 \qquad (2a)$$

and

$$\frac{d^2}{dt^2}v + \Omega^2 \cdot v = 0 \qquad (2b)$$

where $\lambda = \Omega \cdot \left(\frac{\rho \cdot A}{E \cdot I_P}\right)^{.5}$

The solution to the ODEs is simple, i.e.:

$$v(t) = A_t \cdot \cos(\Omega \cdot t) + B_t \cdot \sin(\Omega \cdot t)$$

(3)

$$\phi(x) = A_x \cdot \cos(\beta \cdot x) + B_x \cdot \sin(\beta \cdot x) + C_x \cdot \cosh(\beta \cdot x) + D_x \cdot \sinh(\beta \cdot x)$$

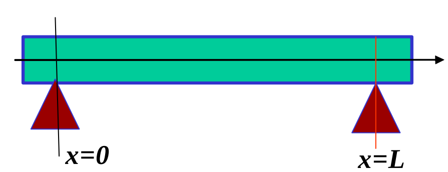
$\beta = \lambda^{0.5}$

Satisfy the boundary conditions.

PINNED ENDs: no lateral displacement and null bending moment

At x=0, L

$u = 0 \qquad \frac{d^2}{dx^2}u = M = 0$



at x=0

$$\phi(0) = A_x \cdot \cos(\beta \cdot 0) + C_x \cdot \cosh(\beta \cdot 0) = 0$$

$$\frac{d^2}{dx^2}\phi = -A_x \cdot \cos(\beta \cdot 0) + C_x \cdot \cosh(\beta \cdot 0) = 0$$

since: $\cos(0) = 1 \quad \cosh(0) = 1$

then $A_x = C_x = 0$ so $\phi(x) = B_x \cdot \sin(\beta \cdot x) + D_x \cdot \sinh(\beta \cdot x)$

at x=L

$$\phi(L) = B_x \cdot \sin(\beta \cdot L) + D_x \cdot \sinh(\beta \cdot L) = 0$$

$$\frac{d^2}{dx^2}\phi = -B_x \cdot \sin(\beta \cdot L) + D_x \cdot \sinh(\beta \cdot L) = 0$$

then $D_x = 0$

MODE SHAPE

$\sin(\beta \cdot L) = 0$ (4) thus $\phi(x) = \sin(\beta \cdot x)$ (5)

(4) is the **characteristic equation** $i := 1 .. n$ $n := 7$

with solution: # of roots

$$\beta_i := \frac{i \cdot \pi}{L}$$

and natural frequencies

$$\Omega := (\beta)^2 \cdot \left(\frac{E \cdot I_p}{\rho \cdot A}\right)^{0.5}$$

$$\beta = \begin{pmatrix} 3.142 \\ 6.283 \\ 9.425 \\ 12.566 \\ 15.708 \\ 18.85 \\ 21.991 \end{pmatrix} \frac{1}{m}$$

$$f := \frac{\Omega}{2 \cdot \pi}$$

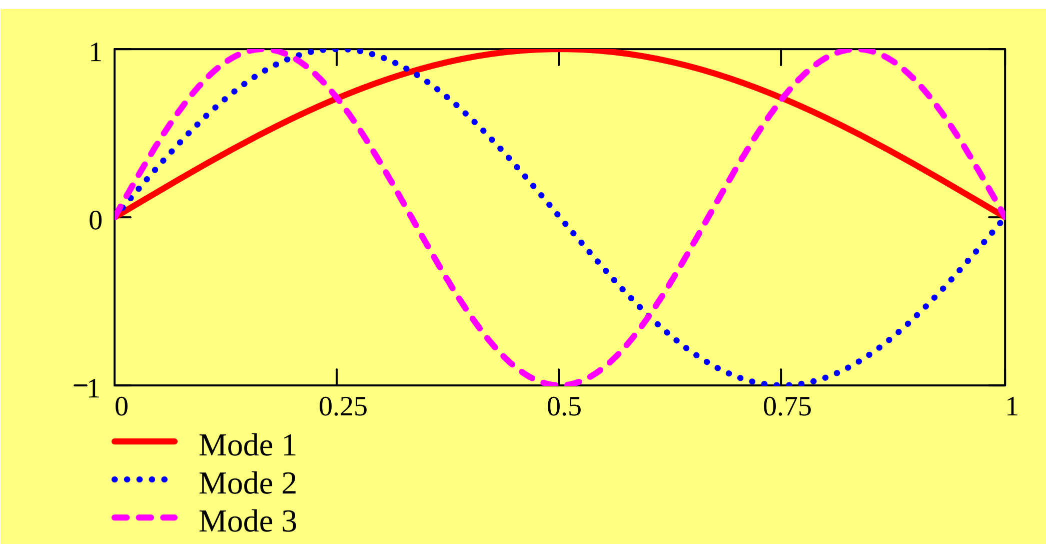
natural frequencies

$$\Omega^T = \begin{pmatrix} 187.413 & 749.65 & 1.687 \times 10^3 & 2.999 \times 10^3 & 4.685 \times 10^3 & 6.747 \times 10^3 & 9.183 \times 10^3 \end{pmatrix} \frac{rad}{s}$$

The **shape functions** are

$\phi1(x) := \sin(\beta_1 \cdot x)$ $\phi2(x) := \sin(\beta_2 \cdot x)$ $\phi3(z) := \sin(\beta_3 \cdot z)$

$$f^T = \begin{pmatrix} 29.828 & 119.311 & 268.449 & 477.242 & 745.691 & 1.074 \times 10^3 & 1.462 \times 10^3 \end{pmatrix} Hz$$



(b) Approximate first natural frequency. Using mode shape $\varphi_a(x) := \sin\left(\pi \cdot \frac{x}{L}\right)$

$\varphi_a(x) := \frac{4x}{L} \cdot \left(1 - \frac{x}{L}\right)$

recall lots of problems worked in class and homeworks, one can easily estimate the equivalent stiffness and mass as

$$M_{eq} := \rho \cdot A \cdot \int_0^L \varphi_a(x)^2 \, dx$$

$$\frac{M_{eq}}{\rho \cdot A \cdot L} = 0.533$$

$$K_{eq} := E \cdot I_P \cdot \int_0^L \left(\frac{d^2}{dx^2}\varphi_a(x)\right)^2 dx$$

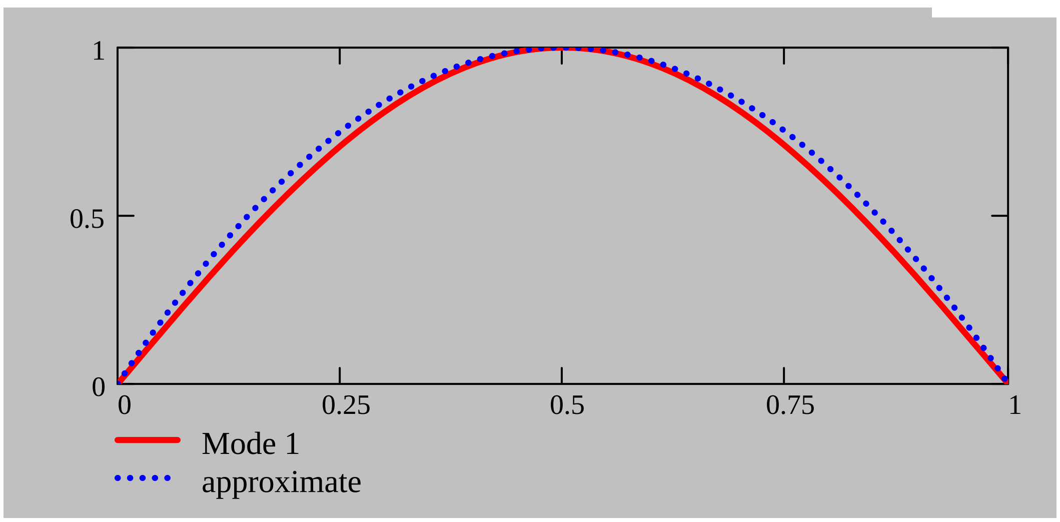
$$\frac{K_{eq}}{\frac{48 \cdot E \cdot I_P}{L^3}} = 1.333$$

and the estimation for **natural frequency** is

$$\Omega_{1a} := \left(\frac{K_{eq}}{M_{eq}}\right)^{.5}$$

$$\Omega_{1a} = 208.013 \frac{rad}{s}$$

$$\Omega_1 = 187.413 \frac{rad}{s}$$ exact



$$\frac{\Omega_{1a}}{\Omega_1} = 1.11$$

(c) Free vibrations response:

Consider the following intitial conditions,

at t=0 $u(x,0) = 0$ **No initial deformation**

$\frac{d}{dt}u = v_o$ **velocity = free fall velocity**

The response for lateral motions of a beam is

.

$$u(x,t) = \sum_{i=1}^{n} \phi_i \cdot \left(A_i \cdot \cos(\Omega_i \cdot t) + B_i \cdot \sin(\Omega_i \cdot t)\right)$$

Since the initial displacement = 0 everywhere, then it follows that $A_i = 0$

and for velocity $\frac{d}{dt}u = v_o = \sum_{i=1}^{\cdot} \phi_i \cdot (B_i \cdot \Omega_i)$

multiplying this equation by ϕj and integrating over the domain gives

$i := 1..n$

"Used orthogonality property of shape functions

$$B_i := \frac{\left(\int_0^L \sin(\beta_i \cdot x)\,dx\right) \cdot v_o}{\int_0^L \sin(\beta_i \cdot x)^2\,dx \cdot \Omega_i}$$

$B_i =$

0.03
0
$1.114 \cdot 10^{-3}$
0
$2.407 \cdot 10^{-4}$
0
$8.772 \cdot 10^{-5}$

m

Calculate time response at various spatial points in beam:

$mm := n - 0$ select number of modes to display results

$n = 7$

$$u(x,t) := \sum_{i=1}^{mm} \sin(\beta_i \cdot x) \cdot \left(B_i \cdot \sin(\Omega_i \cdot t)\right)$$ **DISPLACEMENT**

$mm = 7$

VELOCITY

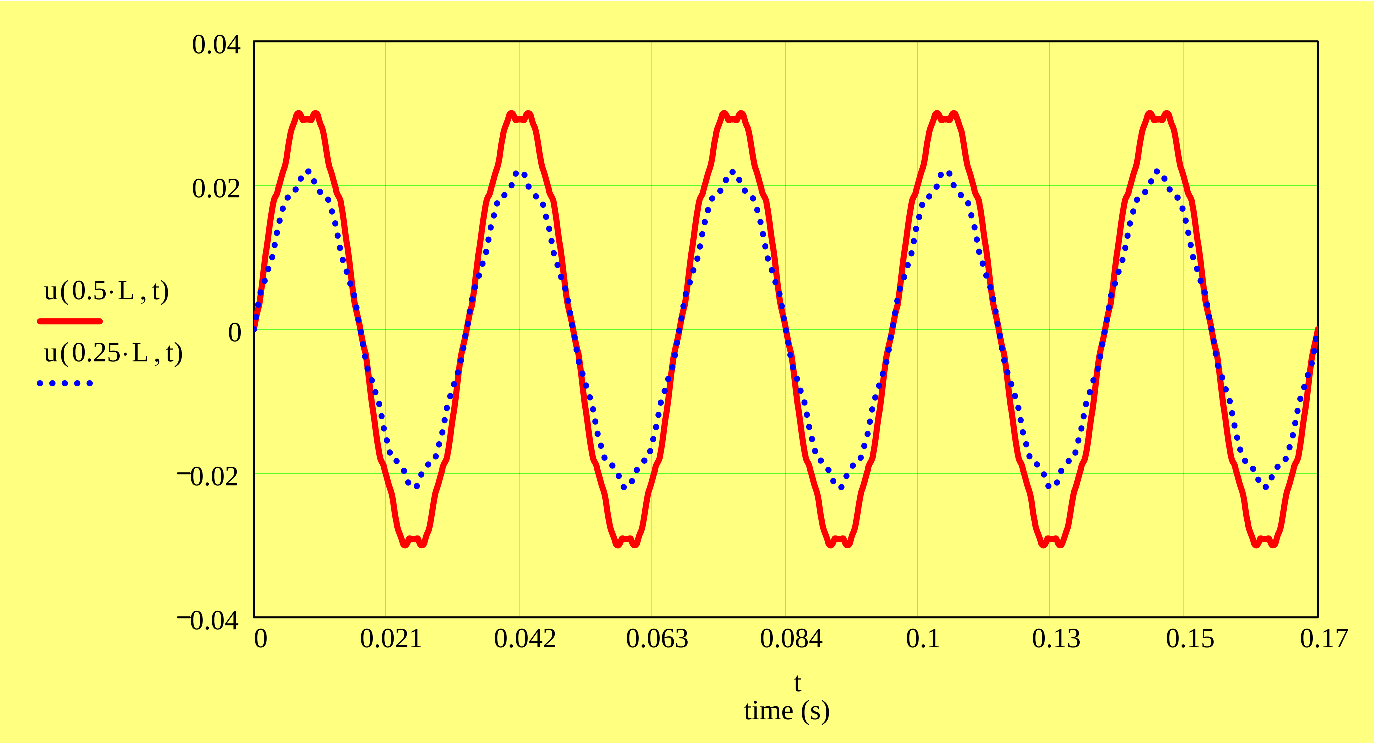
$$vel(x,t) := \sum_{i=1}^{mm} \sin(\beta_i \cdot x) \cdot \Omega_i \cdot \left(B_i \cdot \cos(\Omega_i \cdot t)\right)$$

$v_o = 4.429 \frac{m}{s}$

$T_n := \frac{1}{f}$ Natural periods

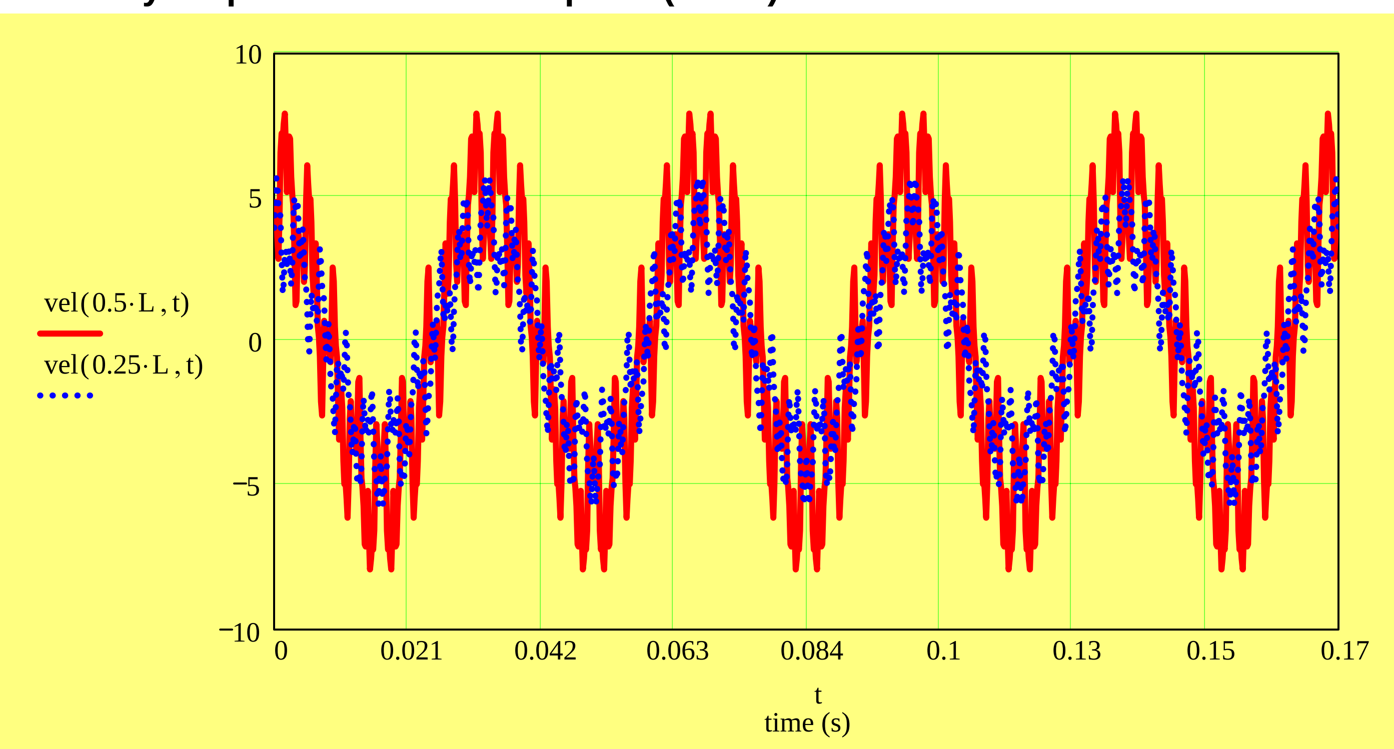
for graph $T_{max} := 5 \cdot T_{n_1}$

Displacement response at beam: midpoint (x=L/2) & x=L/4



$$T_n^{\ T} = \left(0.034 \quad 8.381 \times 10^{-3} \quad 3.725 \times 10^{-3} \quad 2.095 \times 10^{-3} \quad 1.341 \times 10^{-3} \quad 9.313 \times 10^{-4} \quad 6.842 \times 10^{-4}\right) s$$

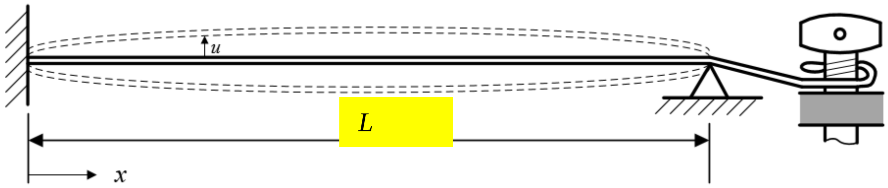
Velocty response at beam: midpoint (x=L/2) & x=L/4



$$XX := \frac{L}{2} \qquad \frac{vel(XX, 0 \cdot s)}{v_o} = 0.922 \qquad v_o = 4.429 \frac{m}{s}$$

Vibrations of a string

This problem aids to understand the tuning process of string musical instruments. The graph shows a simple model of a taut string fixed at both ends.



The string vertical displacement $u(x,t)$ is described by:

$$T\frac{\partial^2 u}{\partial x^2} = \gamma \frac{\partial^2 u}{\partial t^2}$$

where γ = 24.5 g/m is the string mass per unit length, L = 0.5 m is the string length, and T is the tension applied to the string. When the string is plucked at its middle, its vibration response is dominantly represented by the first mode shape at the first natural frequency f_1, (see the dotted lines for a sketch). Then, the sound frequency components radiating from the string are dominant with frequency (f_1).

The tonal frequencies of the strings in a violin are G3 = 196 Hz, D4 = 293.7 Hz, A4 = 440 Hz, and E5 = 659.3 Hz.

Assume the strings are made of the same material (steel). ρ=7800 kg/m^3

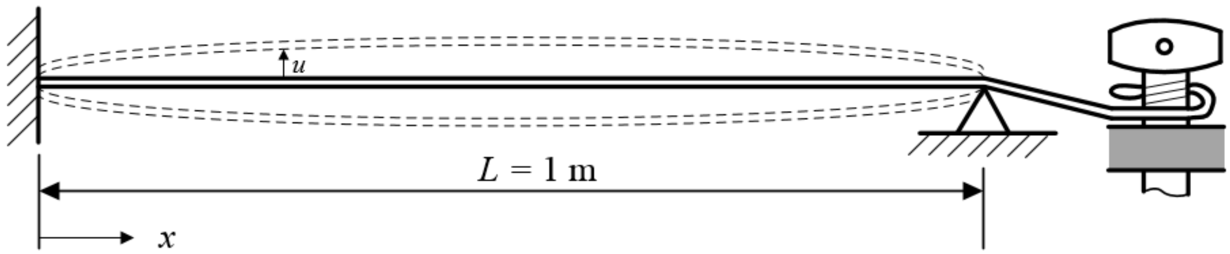
Questions:

a) Find the relation between the frequency (f_1), the string length (L), the mass per unit length (γ), and the tension (T).

b) Assuming all strings have the same diameter, find the tensions T to tune each string. Find also the stresses and elastic deformations.

c) Find how a tonal frequency scales with the diameter of a string. Using the tension found in (b) for G3, determine the strings' diameter and elastic deformation.

Vibrations of a string



ORIGIN := 1

The string vertical displacement $u(x,t)$ is described by

$$T\frac{\partial^2 u}{\partial x^2} = \gamma \frac{\partial^2 u}{\partial t^2} \quad (0)$$

where T is the tension in the string and γ is the mass per unit length.

PHYSICAL Parameters for a steel string:

$E := 30\cdot 10^9 \cdot \frac{N}{m^2}$ $\gamma := 0.0245\cdot\frac{kg}{m}$ $L := 0.5\cdot m$

length of string

(a) natural frequencies & mode shapes

using separation of variables, $\Theta(x,t) = \phi(x)\cdot v(t)$ (1)

Substitute into the field Eq. (0) to obtain the following two ODEs:

$$\frac{d^2}{dx^2}\phi + \lambda^2\cdot\phi = 0 \quad (2a)$$

where $\lambda = \Omega\cdot\left(\frac{\gamma}{T}\right)^{.5}$

and

$$\frac{d^2}{dt^2}v + \Omega^2\cdot v = 0 \quad (2b)$$

Let $c_o = \left(\frac{T}{\gamma}\right)^{0.5}$

a characteristic speed [m/s]

The solution to the ODEs is simple, i.e.:

$$v(t) = A_t\cdot\cos(\Omega\cdot t) + B_t\cdot\sin(\Omega\cdot t)$$

(3)

$$\phi(x) = A_x\cdot\cos(\lambda\cdot x) + B_x\cdot\sin(\lambda\cdot x)$$

Satisfy the boundary conditions. At x=0, Θ(0,t)=0 (**fixed end - no displacement**). Thus

$\phi(0) = A_x\cdot\cos(\lambda\cdot 0) + B_x\cdot\sin(\lambda\cdot 0)$

$\phi(0) = A_x\cdot 1 + B_x\cdot 0$ Then: $A_x = 0$

and $\phi(x) = \sin(\lambda \cdot x)$ (4) is the equation for the shape funcion.

At the right end, x=L, the string is also fixed. Thus

at x=L $\phi(L) = \sin(\lambda \cdot L) = 0$ (5a)

Set $T := 4000 \cdot N$ example

which is the **characteristic equation.** The roots are $\lambda_j := \frac{j \cdot \pi}{L}$

And thus, the **natural frequencies** are $\Omega_j = \lambda_j \cdot c_o$

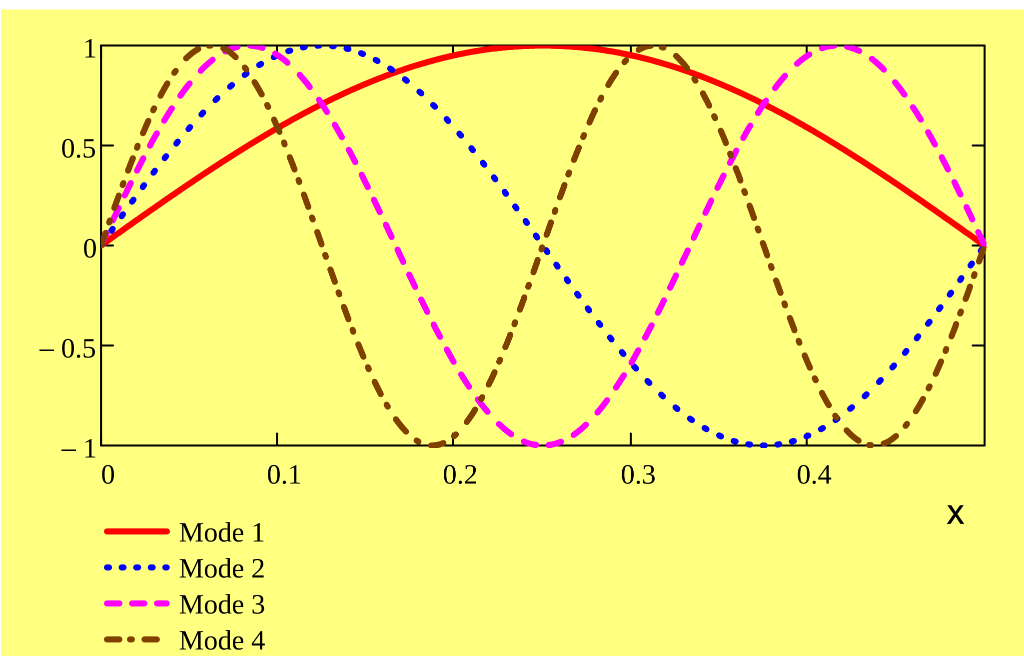
$$\Omega_j := \frac{j \cdot \pi}{L} \cdot \left(\frac{T}{\gamma}\right)^{.5}$$ rad/s

$$\Omega^T = \left(2.539 \times 10^3 \quad 5.078 \times 10^3 \quad 7.616 \times 10^3 \quad 1.016 \times 10^4\right) \frac{rad}{s}$$

The shape functions are

$$\phi 1(x) := \sin\left(x \cdot \frac{\pi}{L}\right) \qquad \phi 2(x) := \sin\left(x \cdot \frac{\pi \cdot 2}{L}\right) \qquad \phi 3(x) := \sin\left(x \cdot \frac{\pi \cdot 3}{L}\right)$$

$$\phi 4(x) := \sin\left(x \cdot \frac{\pi \cdot 4}{L}\right)$$ etc



(b) Find the relation between the frequency (f_1), the string length (L), the mass per unit length (γ), and the tension (T).

The **natural frequencies** in Hz (cycles/s] are $f = \Omega \cdot \frac{1}{2 \cdot \pi}$ $\quad f_j := \frac{j}{2L} \cdot \left(\frac{T}{\gamma}\right)^{.5}$ Hz

The tonal frequencies for the strings in a VIOLIN are

$$\begin{pmatrix} G3 \\ D4 \\ A4 \\ E5 \end{pmatrix} \quad f_{tone} := \begin{pmatrix} 196 \\ 296.7 \\ 440 \\ 659.3 \end{pmatrix} \cdot Hz \quad \begin{pmatrix} Sol_3 \\ Re_4 \\ La_4 \\ Mi_5 \end{pmatrix}$$

$$T = (f \cdot 2 \cdot L)^2 \cdot \gamma$$

$k := 1..4$ (four strings)

considering the same density/length for all strings, the necessary tuning tension in a string is

$$T_k := \left(2 \cdot L \cdot f_{tone_k}\right)^2 \cdot \gamma$$

$$T = \begin{pmatrix} 941.192 \\ 2.157 \times 10^3 \\ 4.743 \times 10^3 \\ 1.065 \times 10^4 \end{pmatrix} N \quad \begin{pmatrix} G3 \\ D4 \\ A4 \\ E5 \end{pmatrix}$$

Let a string diameter $d = 2\,mm$ (assumption as everyone knows the strings have different diameters)

The **stresses** in the strings are $\sigma := \frac{T}{\left(\pi \cdot \frac{d^2}{4}\right)}$ and the **strains** $\varepsilon := \frac{\sigma}{E}$

$$\sigma^T = \left(2.996 \times 10^8 \quad 6.865 \times 10^8 \quad 1.51 \times 10^9 \quad 3.39 \times 10^9\right) Pa$$

$$\varepsilon^T = \left(9.986 \times 10^{-3} \quad 0.023 \quad 0.05 \quad 0.113\right)$$

$d = 2\,mm$

$$\varepsilon^T \cdot L = (4.993 \quad 11.442 \quad 25.163 \quad 56.498)\,mm$$

Deformation for the strings are large, in particular for one with highest tonal frequency.

Recall $L = 500\,mm$

Note: do realize the vibrations of a string is dimensionally and physically equivalent to the axial vibrations of an elastic bar and the torsional vibrations of a rod

(c) Using the tension found in (b)=G3, and assuming the strings are made of the same material (steel), find how a tonal frequency scales with the diameter of each string

Note that a string mass/unit length $\gamma = \frac{\rho \cdot A \cdot L}{L} = \rho \cdot \frac{\pi \cdot d^2}{4}$

where ρ is the material density and d is the diameter of a string. Hence, the tonal frequency as a function of the string diameter equals

$$f_{tone} = \frac{1}{2L} \cdot \left(\frac{T}{\gamma}\right)^{.5} = \frac{1}{2L} \cdot \left(\frac{4T}{\rho \cdot \pi \cdot d^2}\right)^{.5}$$

$$f_{tone} = \frac{1}{L \cdot d} \cdot \left(\frac{T}{\rho \cdot \pi}\right)^{.5}$$

Set $T\_ := T_1$ Tension for sound G3 $T\_ = 941.192\,\mathrm{N}$

Hence, for a given tension, a string's diameter is inversely proportional to the tonal frequency

$$d_k := \frac{1}{L \cdot f_{tone_k}} \cdot \left(\frac{T\_}{\rho \cdot \pi}\right)^{.5}$$

$$d = \begin{pmatrix} 2 \\ 1.321 \\ 0.891 \\ 0.595 \end{pmatrix} \mathrm{mm} \qquad \begin{pmatrix} G3 \\ D4 \\ A4 \\ E5 \end{pmatrix}$$

With deformation for each string

$$\delta_k := \frac{T\_}{\left[\pi \cdot \frac{(d_k)^2}{4}\right]} \cdot \frac{L}{E}$$

$$\delta^T = (4.994 \;\; 11.444 \;\; 25.168 \;\; 56.508)\,\mathrm{mm}$$

$L = 0.5\,\mathrm{m}$

$E = 3 \times 10^{10}\,\mathrm{Pa}$

$$\frac{\delta^T}{L} = \left(9.988 \times 10^{-3} \;\; 0.023 \;\; 0.05 \;\; 0.113\right)$$

(c) Stresses and strains

The **stresses** in the taut strings are $\sigma = \frac{T}{A} = \left(2 \cdot L \cdot f_{tone}\right)^2 \cdot \rho$

since $\gamma = \rho \cdot A$

and the axial **strains** $\varepsilon = \frac{\sigma}{E} = \left(2 \cdot L \cdot f_{tone}\right)^2 \cdot \left(\frac{\rho}{E}\right)$ **NOT a function of the string diameter**

with elastic deformation $\delta = \varepsilon \cdot L = \left(2 \cdot f_{tone}\right)^2 \cdot \left(\frac{\rho}{E}\right) \cdot L^3$ proportional to L^3 and tone_freq^2