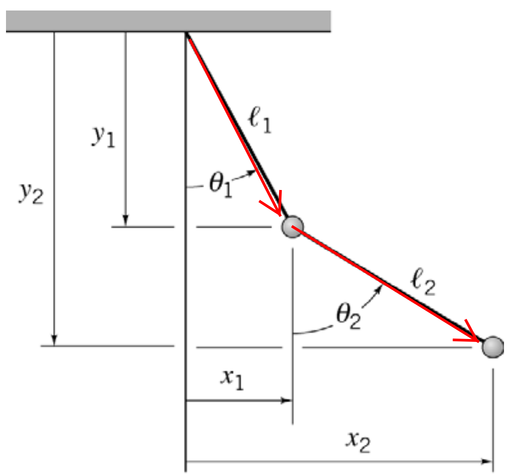
Lecture 13. LAGRANGE'S EQUATIONS OF MOTION

Lagrange developed an alternative approach to deriving equation of motion to Newtons's $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ force differential equation approach. A review of Lagrange's development is the subject of this lecture



The double pendulum has two degrees of freedom; e.g., θ_1, θ_2 could be used to completely define the positions of the two masses. θ_1, θ_2 are *generalized coordinates* satisfying the following definition:

Generalized = Coordinates *The least number of coordinates that are required to define the position and location of a system of particles or rigid bodies.*

Choosing to use more coordinates than the minimum number of coordinates will result in relationships between the coordinates; e.g., using x_1, x_2, y_1, y_2 yields

lengths l1 and l2 are constant

$$x_1^2 + y_1^2 = l_1^2 \; , \quad x_2^2 + y_2^2 = l_2^2 \; . \tag{6.1}$$

The Cartesian coordinates x_1, x_2, y_1, y_2 can be defined in terms of the generalized coordinates θ_1, θ_2 as

$$\begin{aligned} x_1 &= l_1 \sin\theta_1 \; , \; y_1 = l_1 \cos\theta_1 \, , \\ x_2 &= l_1 \sin\theta_1 + l_2 \sin\theta_2 \; , \quad y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2 \; . \end{aligned} \tag{6.2}$$

We are going to develop Lagrange's equations of motion for a system of n particles with k degrees of freedom. For the double-pendulum, with θ_1, θ_2 as generalized coordinates the number of degrees of freedom equaled the number of particles. Using x_1, x_2, y_1, y_2 as generalized coordinates would yield four coordinates for two degrees of freedom. In general, $k \le n$.

The kinetic energy for our system of particles is

$$T=\frac{1}{2}\sum_{j=1}^{n} m_j v_j^2 = \frac{1}{2}\sum_{j=1}^{n} m_j(\dot{x}_j^2+\dot{y}_j^2+\dot{z}_j^2) \ , \qquad (6.3)$$

where $\boldsymbol{v_j}=\boldsymbol{i}\dot{x}_j+\boldsymbol{j}\dot{y}_j+\boldsymbol{k}\dot{x}_j$ is the velocity of mass m_j with respect to an inertial coordinate system. For the double-pendulum example, this equation is

$$T=\frac{m_1}{2}(\dot{x}_1^2+\dot{y}_1^2)+\frac{m_2}{2}(\dot{x}_2^2+\dot{y}_2^2) \ . \qquad (6.4)$$

Returning to the general situation, the n Cartesian coordinates, $(x_i,y_i,z_i;i=1,n)$ can be stated in terms of the k generalized coordinates $(q_i;i=1,k)$ as

$$x_j=\varphi_{j1}(q_1,q_2,\ldots\ldots q_k),$$
$$y_j=\varphi_{j2}(q_1,q_2,\ldots\ldots q_k),$$
$$z_j=\varphi_{j3}(q_1,q_2,\ldots\ldots q_k) \ .$$

holonomic constraints: not explicit functions of time

Eqs.(6.2) provide these expressions for the double-pendulum example with (θ_1,θ_2) serving as generalized coordinates.

Differentiating Eqs.(6.5) defines the Cartesian velocity components as

velocities:

$$\dot{x}_j=\sum_{i=1}^{k}\frac{\partial\varphi_{j1}}{\partial q_i}\dot{q}_i \ ,$$
$$\dot{y}_j=\sum_{i=1}^{k}\frac{\partial\varphi_{j2}}{\partial q_i}\dot{q}_i, \qquad (6.6)$$
$$\dot{z}_j=\sum_{i=1}^{k}\frac{\partial\varphi_{j3}}{\partial q_i}\dot{q}_i \ .$$

where $\dot{q}_i$ is the ith generalized velocity. For the double-pendulum example, the two generalized velocities are $(\dot{\theta}_1,\dot{\theta}_2)$.

Outline of the Derivation

The physical expression that we will use in developing Lagrange's equation is

$$dT=dW \ \Rightarrow \ dT-dW=0 \ , \qquad (6.7)$$

which states that, for a general system of particles, the change in system kinetic energy equals the change in work due to all forces (conservative and nonconservative). In the following developments we will obtain expressions for dT and dW of the form:

$$dT=A_1(q_i,\dot{q}_i)dq_1+A_2(q_i,\dot{q}_i)dq_2+\ldots\ldots+A_k(q_i,\dot{q}_i)dq_k \ ;$$
$$dW=B_1(q_i,\dot{q}_i)dq_1+B_2(q_i,\dot{q}_i)dq_2+\ldots\ldots+B_k(q_i,\dot{q}_i)dq_k \ . \qquad (6.8)$$

Substituting these expressions into $dT - dW = 0$ gives

$$[A_1(q_i,\dot{q}_i) - B_1(q_i,\dot{q}_i)]dq_1 + [A_2(q_i,\dot{q}_i) - B_2(q_i,\dot{q}_i)]dq_2 + \ldots + [A_k(q_i,\dot{q}_i) - B_k(q_i,\dot{q}_i)]dq_k = 0 \ . \qquad (6.9)$$

Because the generalized coordinates are linearly independent, each of the coefficients must equal zero, giving the k differential equations of motion

$$A_j(q_i,\dot{q}_i) - B_j(q_i,\dot{q}_i) = 0 \ ; \ j,i = 1,\ldots k \ . \qquad (6.10)$$

Most of the work in the development involves getting the expressions for dT and dW in Eqs.(6.8) and (6.9), respectively.

Derivation

Hoping that you now understand the variables and the ideas, we will proceed with the derivation. Substituting From Eq.(6.6) into Eq.(6.5) gives

$$T = \frac{1}{2}\sum_{j=1}^{n} m_j[(\sum_{i=1}^{k}\frac{\partial\varphi_{j1}}{\partial q_i}\dot{q}_i)^2 + (\sum_{i=1}^{k}\frac{\partial\varphi_{j2}}{\partial q_i}\dot{q}_i)^2 + (\sum_{i=1}^{k}\frac{\partial\varphi_{j3}}{\partial q_i}\dot{q}_i)^2] \ . \qquad (6.11)$$

Note that this expression for the kinetic energy is a function only of the generalized coordinates and the generalized velocities; i.e.,

$$T = T(q_i,\dot{q}_i) \ . \qquad (6.12)$$

Taking the partial derivative of T with respect to each

generalized velocity gives the following k equations

$$\frac{\partial T}{\partial \dot{q}_m} = \sum_{j=1}^{n} m_j[(\sum_{i=1}^{k}\frac{\partial\varphi_{j1}}{\partial q_i}\dot{q}_i)\frac{\partial\varphi_{j1}}{\partial q_m} + (\sum_{i=1}^{k}\frac{\partial\varphi_{j2}}{\partial q_i}\dot{q}_i)\frac{\partial\varphi_{j2}}{\partial q_m} + (\sum_{i=1}^{k}\frac{\partial\varphi_{j3}}{\partial q_i}\dot{q}_i)\frac{\partial\varphi_{j3}}{\partial q_m}] ; \ m = 1,k \ .$$

Multiplying each expression by the appropriate $\dot{q}_m$ gives

$$\dot{q}_m\frac{\partial T}{\partial \dot{q}_m} = \sum_{j=1}^{n} m_j[(\sum_{i=1}^{k}\frac{\partial\varphi_{j1}}{\partial q_i}\dot{q}_i)\frac{\partial\varphi_{j1}}{\partial q_m}\dot{q}_m + (\sum_{i=1}^{k}\frac{\partial\varphi_{j2}}{\partial q_i}\dot{q}_i)\frac{\partial\varphi_{j2}}{\partial q_m}\dot{q}_m + (\sum_{i=1}^{k}\frac{\partial\varphi_{j3}}{\partial q_i}\dot{q}_i)\frac{\partial\varphi_{j3}}{\partial q_m}\dot{q}_m \ ] ; \ m = 1,k \ .$$

use Kroneker delta fn

Summing these k equations gives

(6.6) y_dot^2

$$\sum_{m=k}^{n}\dot{q}_m\frac{\partial T}{\partial \dot{q}_m} = \sum_{j=1}^{n} m_j[(\sum_{i=1}^{k}\frac{\partial\varphi_{j1}}{\partial q_i}\dot{q}_i)^2 + (\sum_{i=1}^{k}\frac{\partial\varphi_{j2}}{\partial q_i}\dot{q}_i)^2 + (\sum_{i=1}^{k}\frac{\partial\varphi_{j3}}{\partial q_i}\dot{q}_i)^2]$$

$$= \sum_{j=1}^{n} m_j(\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2) = 2T \ .$$

The abrupt simplification of this result follows from Eq.(6.6) that defines the Cartesian-coordinate velocities in terms of generalized velocities. Reversing this equation gives

$$2T=\sum_{m=1}^{k}\dot{q}_m\frac{\partial T}{\partial \dot{q}_m} . \tag{6.13}$$

Differentiating with respect to time yields

$$2\frac{dT}{dt}=\Sigma_{m=1,k}\left[\ddot{q}_m\frac{\partial T}{\partial \dot{q}_m}+\dot{q}_m\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_m}\right)\right] . \tag{6.14}$$

We can use the functional definition of Eq.(6.12), $T=T(q_i,\dot{q}_i)$ to obtain the following alternative statement for dT/dt

$$\frac{dT}{dt}=\sum_{m=1}^{k}\left[\frac{\partial T}{\partial q_m}\dot{q}_m+\frac{\partial T}{\partial \dot{q}_m}\ddot{q}_m\right] . \tag{6.15}$$

Subtracting Eq.(6.15) from (6.14) gives

$$\frac{dT}{dt}=\sum_{j=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_m}\right)-\frac{\partial T}{\partial q_m}\right]\dot{q}_m .$$

Hence,

$$dT=\sum_{m=1}^{k}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_m}\right)-\frac{\partial T}{\partial q_m}\right]q_m . \tag{6.16}$$

This is the desired final expression for dT cited earlier in Eq.(6.8).

Moving onwards to develop the comparable expression for dW, we can state

WORK

$$dW=\sum_{j=1}^{n}\boldsymbol{f_j}\bullet\boldsymbol{ds}=\sum_{j=1}^{n}(f_{xj}dx_j+f_{yj}dy_j+f_{xj}dz_j)$$

$$=\sum_{j=1}^{n}\left[f_{xj}\left(\sum_{i=1}^{k}\frac{\partial\varphi_{j1}}{\partial q_i}dq_i\right)+f_{yj}\left(\sum_{i=1}^{k}\frac{\partial\varphi_{j2}}{\partial q_i}dq_i\right)+f_{zj}\left(\sum_{i=1}^{k}\frac{\partial\varphi_{j3}}{\partial q_i}dq_i\right)\right] .$$

Reversing the order of summation gives the final desired result

$$dWork=\sum_{i=1}^{k}\left[\left(\sum_{j=1}^{n}f_{xj}\frac{\partial\varphi_{j1}}{\partial q_i}\right)+\left(\sum_{j=1}^{n}f_{yj}\frac{\partial\varphi_{j2}}{\partial q_i}\right)+\left(\sum_{j=1}^{n}f_{zj}\frac{\partial\varphi_{j3}}{\partial q_i}\right)\right]dq_i$$

$$=\sum_{i=1}^{k}Q_i dq_i , \tag{6.17}$$

where the Q_i's are the generalized force terms defined by

$$Q_i=\left(\sum_{j=1}^{n}f_{xj}\frac{\partial\varphi_{j1}}{\partial q_i}\right)+\left(\sum_{j=1}^{n}f_{yj}\frac{\partial\varphi_{j2}}{\partial q_i}\right)+\left(\sum_{j=1}^{n}f_{zj}\frac{\partial\varphi_{j3}}{\partial q_i}\right) .$$

Recall that several of the one-degree-of-freedom problems of subsections 3.6b and 5.7c involved generalized forces.

Eqs.(6.16) and (6.18) provide the requisite definitions for dT and dW for Eq.(6.8); hence

6.17

$$\sum_{m=1}^{k} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_m}\right) - \frac{\partial T}{\partial q_m} - Q_m\right] dq_m = 0 \ . \tag{6.18}$$

Because of the q_m's linear independence , all of the dq_m coefficients must be zero, and the Euler-Lagrange differential equations of motion are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_m}\right) - \frac{\partial T}{\partial q_m} = Q_m \ ; \ \ m = 1,2,\ldots k \ . \tag{6.19}$$

This form of the equations can be made more useful by separating the generalized forces into conservative and nonconservative terms as

$$Q_m = Q_{m\,cons} + Q_{mn.c.} = -\frac{\partial V(q_i)}{\partial q_m} + Q_{mn.c.} \ .$$

Remember that a conservative force can be expressed as the gradient of the potential-energy function *V*. Note also that *V* is a function only of the generalized coordinates, not the generalized velocities $\dot{q}_i$. With this definition, Eq.(6.19) can be stated

$$\frac{d}{dt}\left[\frac{\partial (T-V)}{\partial \dot{q}_m}\right] - \frac{\partial (T-V)}{\partial q_m} = Q_{mn.c.} \ ; \ \ m = 1,2,\ldots k \ ,$$

where, $\partial V / \partial \dot{q}_m = 0$ because *V* is only a function of the coordinates q_m . Finally,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) - \frac{\partial L}{\partial q_m} = Q_{mn.c.} \ ; \ \ m = 1,2,\ldots k \ , \tag{6.20}$$

where the "Lagrangian function" *L* is defined as

$$L = T - V \ . \tag{6.21}$$

"Dynamicists" occasionally argue the issue: Is the variational approach more or less fundamental than the Newtonian approach to dynamics? Note in developing the present equations that the physics of this development starts with $dT = dW$, an integrated form of $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$, and, Newton's approach is taken as the starting point.

Applying Lagrange's Equation of Motion to Problems Without Kinematic Constraints

The contents of this section will demonstrate the application of Eqs.(6.20) in developing equations of motion for systems of particles and rigid bodies. The examples considered in this section have generalized coordinates implying no constraint relationships (equations) between the coordinates.

Two-mass Vibration Example

The engineering-analysis task associated with this problem is: *Apply the Euler-Lagrange Eqs.(6.20) to derive the governing equations of motion*. The logical generalized coordinates to use for the task are x_1, x_2 with their associated generalized velocities $\dot{x}_1, \dot{x}_2$. The kinetic and potential energy functions are defined in terms of these variables by

$$T = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2} \quad , \quad V = \frac{k_1}{2} x_1^2 + \frac{k_2}{2}(x_2 - x_1)^2 + k_3 \frac{x_3^2}{2} \;;$$

hence,

$$L = \left(\frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2}\right) - \frac{k_1}{2} x_1^2 - \frac{k_2}{2}(x_2 - x_1)^2 - k_3 \frac{x_3^2}{2} \; .$$

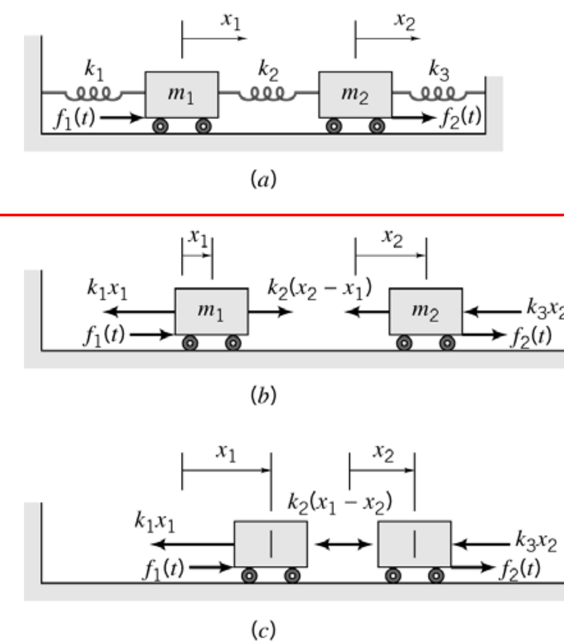


Figure 6.2 Coupled, two-degree of freedom spring-mass system.

The differential work due to nonconservative forces is

$$dW_{n.c.} = \boldsymbol{i} f_1 \bullet \boldsymbol{i} dx_1 + \boldsymbol{i} f_2 \bullet \boldsymbol{i} dx_2 = f_1 dx_1 + f_2 dx_2 \; .$$

From Eqs.(6.18), the generalized forces associated with dx_1 and dx_2 are, respectively, f_1 and f_2. The equations of motion are, accordingly:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}=Q_1 \Rightarrow \frac{d}{dt}(m_1\dot{x}_1)-[-k_1x_1+k_2(x_2-x_1)]=f_1$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\frac{\partial L}{\partial x_2}=Q_2 \Rightarrow \frac{d}{dt}(m_1\dot{x}_2)-[-k_2(x_2-x_1)-k_3x_2]=f_2\ , \quad (6.22)$$

and the final differential equations of motion are:

$$m_1\ddot{x}_1+(k_1+k_2)x_1-k_2x_2=f_1$$
$$m_2\ddot{x}_2-k_1x_1+(k_2+k_3)x_2=f_2\ . \quad (6.23)$$

As expected, these equations of motion coincide with Eqs.(3.101) obtained via $\Sigma \boldsymbol{f}=m\ddot{\boldsymbol{r}}$.

However, this apparent advantage of the variational approach disappears for the two-mass example of figure 3.44 which includes damping. Eq.(3.103) provides the differential equations of motion (from $\Sigma \boldsymbol{f}=m\ddot{\boldsymbol{r}}$) and can be restated as:

$$m_1\ddot{x}_1+(k_1+k_2)x_1-k_2x_2=f_1(t)-(c_1+c_2)\dot{x}_1+c_2\dot{x}_2=Q_1$$
$$m_2\ddot{x}_2-k_2x_1+(k_2+k_3)x_2=f_2(t)+c_2\dot{x}_1-(c_2+c_3)\dot{x}_2=Q_2\ . \quad (6.24)$$

The terms on the right Q_1, Q_2 are the net nonconservative forces acting on the two masses. Substituting these values into Eq.(6.20) to define the differential work due to nonconservative forces gives

$$dW_{n.c.}=\boldsymbol{i}Q_1\bullet\boldsymbol{i}dx_1+\boldsymbol{i}Q_2\bullet\boldsymbol{i}dx_2=Q_1dx_1+Q_2dx_2\ .$$

Substituting for Q_1, Q_2 from Eq.(6.24) will yield the correct differential equations; however, Q_1, Q_2 are obtained from the free-body diagrams of figure 3.43b, and their development constitutes most of the work in arriving at the equations of motion, Eqs.(3.103). Since a central advantage of the variational approach is avoidance of free-body diagrams, energy-dissipation forces reduce some of the advantages of the Euler-Lagrange equations.

Double-Pendulum Example

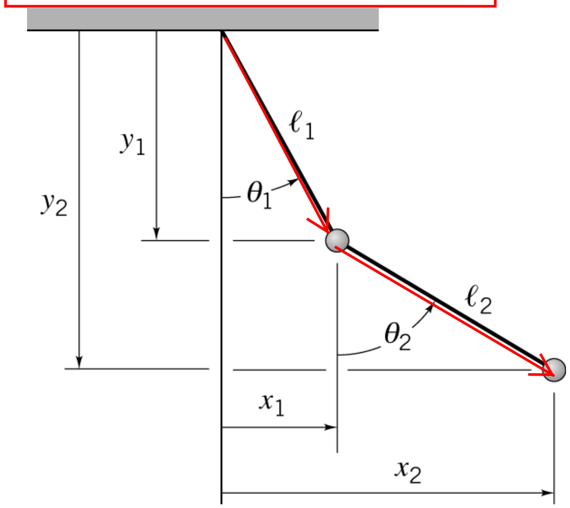
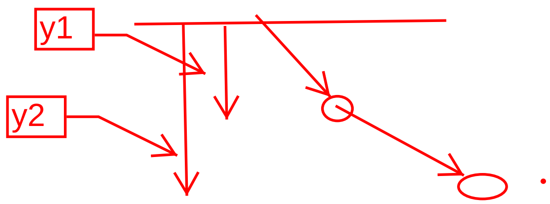


Figure 3.25 Double pendulum.

Develop the governing equations of motion using the Euler-

Lagrange equations.



The angles θ_1, θ_2 will be used as generalized coordinates. From Eqs.(6.2),

$$x_1 = l_1 \sin\theta_1 \ , \ y_1 = l_1 \cos\theta_1 ,$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \ , \quad y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2 \ . \qquad (6.2)$$

Differentiating fives:

$$\dot{x}_1 = l_1 \cos\theta_1 \dot{\theta}_1 \ , \ \dot{y}_1 = -l_1 \sin\theta_1 \dot{\theta}_1$$

$$\dot{x}_2 = l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2, \ , \quad \dot{y}_2 = -l_1 \sin\theta_1 \dot{\theta}_1 - l_2 \sin\theta_2 \dot{\theta}_2 \ .$$

Hence, the kinetic energy function is

$$T = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{m_1}{2}(l_1\dot{\theta}_1)^2 + \frac{m_2}{2}[(l_1\dot{\theta}_1)^2 + (l_2\dot{\theta}_2)^2 + 2 l_1 l_2 \dot{\theta}_1 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)\dot{\theta}_2]$$

$$= \frac{m_1}{2}(l_1\dot{\theta}_1)^2 + \frac{m_2}{2}[(l_1\dot{\theta}_1)^2 + (l_2\dot{\theta}_2)^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)] \ .$$

Using a horizontal line through the upper pivot point as datum leads to the following definition for the potential-energy function

$$V = -w_1 y_1 - w_2 y_2 = = -w_1 l_1 \cos\theta_1 - w_2 (l_1 \cos\theta_1 + l_2 \cos\theta_2) \ .$$

Hence, the Lagrangian is

$$L = \frac{(m_1 + m_2)}{2} l_1^2 \dot{\theta}_1^2 + \frac{m_2}{2}[(l_2\dot{\theta}_2)^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)]$$

$$+ [w_1 l_1 \cos\theta_1 + w_2 (l_1 \cos\theta_1 + l_2 \cos\theta_2)] \ .$$

For the coordinates selected, the Euler-Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \ , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0 \ . \qquad (6.25)$$

The partial derivatives with respect to generalized velocities are:

$$\frac{\partial L}{\partial \dot{\theta}_1} = l_1^2 (m_1 + m_2)\dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \ .$$

The derivatives of these terms with respect to time gives:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) = l_1^2(m_1+m_2)\ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1-\theta_2)$$
$$- m_2 l_1 l_2 \dot\theta_2 \sin(\theta_1-\theta_2)(\dot\theta_1-\dot\theta_2)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1-\theta_2) \tag{6.26}$$
$$- m_2 l_1 l_2 \dot\theta_1 \sin(\theta_1-\theta_2)(\dot\theta_1-\dot\theta_2) \ .$$

Note the last terms in these expressions. Continuing with the partial differentiation yields:

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2) - (w_1+w_2) l_1 \sin\theta_1$$
$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2) - w_2 l_2 \sin\theta_2 \ . \tag{6.27}$$

Substituting these partial derivatives into Eq.(6.25) defines the governing equation of motion for θ_1 and θ_2 as:

$$(m_1+m_2) l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_2 \sin(\theta_1-\theta_2)(\dot\theta_1-\dot\theta_2)$$
$$+ m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2) + (w_1+w_2) l_1 \sin\theta_1 = 0$$
$$m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_1 \sin(\theta_1-\theta_2)(\dot\theta_1-\dot\theta_2)$$
$$- m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2) + w_2 l_2 \sin\theta_2 = 0 \ .$$

These equations reduce to:

$$(m_1+m_2) l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_2^2 \sin(\theta_1-\theta_2$$
$$+ (w_1+w_2) l_1 \sin\theta_1 = 0$$
$$m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_1^2 \sin(\theta_1-\theta_2) \tag{6.28}$$
$$+ w_2 l_2 \sin\theta_2 = 0 \ ,$$

and coincide with Eqs.(3.137) that were derived from $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$. Note that these equations were developed without recourse to free-body diagrams, and only velocities were required in the kinematics development.

Please go back and review the Newtonian development of these equations in subsection 3.5b. Following the statement of Eq.(3.109), there are three equations in the three unknowns $(T_{c2}, \ddot\theta_1, \ddot\theta_2)$. Note particularly T_{c2}, the tension in the lower cord and its presence in the free-body diagram of figure 3.43. In a Newtonian development, T_{c2} is of obvious importance in accounting for the forces acting on the two masses. However, in the Lagrangian formulation, T_{c2} does no work; *accordingly it does not appear in the Lagrangian development.* The "nonappearance" of reaction forces in the Lagrangian development is a major factor in their utility. You don't need to spend time working through algebra to eliminate them, because they never appear in the first place.