Lecture 14. APPLYING LAGRANGE'S EQUATION OF MOTION TO EXAMPLES WITH GENERALIZED COORDINATES (NO KINEMATIC CONSTRAINTS).

Coupled Cart/Pendulum

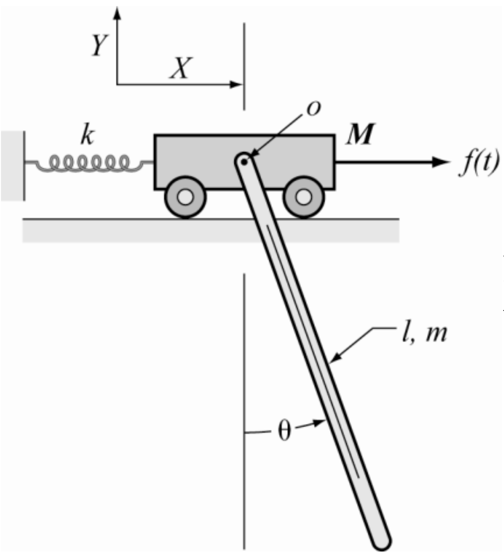

Figure 6.3 Translating cart with an attached pendulum

This system has the two coordinates X, θ and two degrees of freedom. Hence, the two coordinates X, θ are the generalized coordinates q_i, and their derivatives $\dot{X}, \dot{\theta}$ are the generalized velocities $\dot{q}_i$ of Lagrange's equations. The following engineering task applies: *Use Lagrange's equations to derive the equations of motion.*

The kinetic energy of the cart is easily calculated as $T_{cart} = M\dot{X}^2/2$.

The kinetic energy of the pendulum follows from the general kinetic energy for planar motion of a rigid body

$$T = \frac{m\,|\dot{\boldsymbol{R}}_g|^2}{2} + \frac{I_g\,\dot{\theta}^2}{2} . \qquad (5.183)$$

where $\dot{\boldsymbol{R}}_g$ is the velocity of the body's mass center with respect to an inertial coordinate system. The pendulum's mass center is located by

$$X_g = X + \frac{l}{2}\sin\theta \ , \ Y_g = \frac{l}{2}\cos\theta \ .$$

Hence

$$\dot{X}_g = \dot{X} + \frac{l}{2}\cos\theta\,\dot{\theta} \ , \ \dot{Y}_g = -\frac{l}{2}\sin\theta\,\dot{\theta} \ ,$$

and

$$T_{pendulum} = \frac{m}{2}[(\dot{X} + \frac{l}{2}\cos\theta\dot{\theta})^2 + (\frac{l}{2}\sin\theta\,\dot{\theta})^2] + \frac{ml^2}{24}\dot{\theta}^2$$
$$= \frac{ml^2}{6}\dot{\theta}^2 + \frac{m}{2}\dot{X}^2 + \frac{ml}{2}\dot{X}\dot{\theta}\cos\theta \ .$$

Hence, the system kinetic energy is

$$T = \frac{ml^2}{6}\dot{\theta}^2 + \frac{(m+M)}{2}\dot{X}^2 + \frac{ml}{2}\dot{X}\dot{\theta} .$$

Using a plane through the pivot point as datum for the gravity potential energy function gives $V_g = -wl/2\sin\theta$. The potential energy of the spring is $V_s = kX^2/2$; hence, the system potential energy is

$$V = -w\frac{l}{2}\cos\theta + \frac{k}{2}X^2 ,$$

and

$$L = T - V = \frac{ml^2}{6}\dot{\theta}^2 + \frac{(m+M)}{2}\dot{X}^2 + \frac{ml}{2}\dot{X}\dot{\theta}\cos\theta + w\frac{l}{2}\cos\theta - \frac{k}{2}X^2 .$$

Proceeding with the Lagrange equations developments, the partial derivatives with respect to generalized velocities are:

$$\frac{\partial L}{\partial \dot{X}} = (m+M)\dot{X} + \frac{ml}{2}\dot{\theta}\cos\theta \ , \quad \frac{\partial L}{\partial \dot{\theta}} = \frac{ml^2}{3}\dot{\theta} + \frac{ml}{2}\dot{X}\cos\theta \ ,$$

and the derivatives of these terms with respect to time are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = (m+M)\ddot{X} + \frac{ml}{2}\ddot{\theta}\cos\theta - \frac{ml}{2}\sin\theta\dot{\theta}^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{ml^2}{3}\ddot{\theta} + \frac{ml}{2}\ddot{X}\cos\theta - \frac{ml}{2}\sin\theta\dot{X}\dot{\theta} \ .$$

Once again, note the last terms in these derivatives.

The partial derivatives of L with respect to the generalized coordinates are

$$\frac{\partial L}{\partial X} = -kX \ , \quad \frac{\partial L}{\partial \theta} = -\frac{ml}{2}\dot{X}\dot{\theta}\sin\theta - \frac{wl}{2}\sin\theta \ .$$

By substitution, the governing equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) - \frac{\partial L}{\partial X} = 0 \Rightarrow (m+M)\ddot{X} + \frac{ml}{2}\ddot{\theta}\cos\theta - \frac{ml}{2}\sin\theta\dot{\theta}^2 + kX = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{ml^2}{3}\ddot{\theta} + \frac{ml}{2}\ddot{X}\cos\theta - \frac{ml}{2}\sin\theta\dot{X}\dot{\theta} + \frac{ml}{2}\dot{X}\dot{\theta}\sin\theta + \frac{wl}{2}\sin\theta = 0 \ . \qquad (6.29)$$

The right-hand terms are zero, because there are no nonconservative forces. Eqs.(6.29) are stated in matrix notation as

$$\begin{bmatrix} \frac{mL^2}{3} & \frac{mL}{2}\cos\theta \\ \frac{mL}{2}\cos\theta & (M+m) \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{X} \end{Bmatrix} = \begin{Bmatrix} -\frac{wL}{2}\sin\theta \\ -kX+\frac{mL^2}{2}\dot{\theta}^2\sin\theta \end{Bmatrix}, (6.30)$$

which coincides with Eq.(5.160) (without the external force of figure 5.38) that we derived earlier from a free-body diagram/Newtonian approach. Again, the results are obtained without recourse to free-body diagrams, and only velocities are required for the kinematics.

Cart-Pendulum Example with External Forces

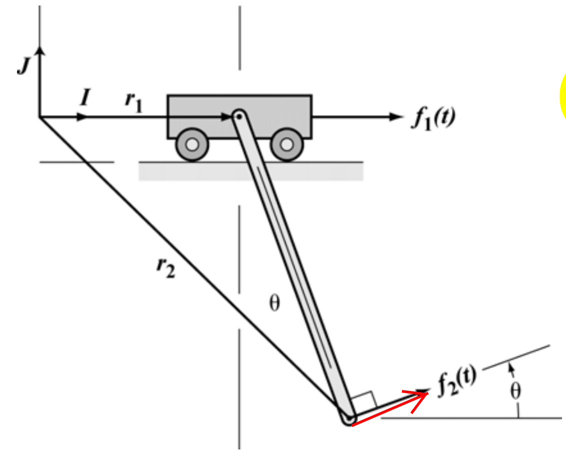

Figure 6.4 Cart-pendulum example with an external force acting on the cart and the end of the pendulum

This example did not include external forces; however, figure 6.4 presents the same system with external forces acting at both the cart and the end of the pendulum. The force on the cart is $\boldsymbol{f_1} = \boldsymbol{I} f_1(t)$, and its point of action is located by the position vector $\boldsymbol{r_1} = \boldsymbol{I} X$. The force acting on the end of the pendulum is $\boldsymbol{f_2} = f_2(t)(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta)$, with a point of action located by the vector $\boldsymbol{r_2} = \boldsymbol{I}(X + l\sin\theta) - \boldsymbol{J} l\cos\theta$. The engineering-analysis task is: *Use Lagrange's equations of motion and develop the equations of motion for the system.* Note that we have already developed the equations of motion in Eqs.(6.29), except for the nonconservative generalized forces. To complete the task, we need to determine the nonconservative generalized forces.

The differential nonconservative force produced by the forces of figure 6.4 due to a differential change in position of the system is

$$dWork_{n.c} = \boldsymbol{f_1} \bullet \boldsymbol{dr_1} + \boldsymbol{f_2} \bullet \boldsymbol{dr_2}$$

$$= \boldsymbol{I} f_1(t) \bullet \boldsymbol{I} dX + f_2(t)(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta) \bullet [\boldsymbol{I}(dX + l\cos\theta\, d\theta) + \boldsymbol{J} l \sin\theta\, d\theta]$$

$$= [f_1(t) + f_2(t)\cos\theta]\, dX + f_2(t)\, l(\cos^2\theta + \sin^2\theta)\, d\theta$$

$$= Q_X dX + Q_\theta d\theta \ .$$

Hence, the generalized force terms are:

$$Q_X = f_1(t) + f_2(t)\cos\theta \ , \quad Q_\theta = f_2(t)\, l$$

Inserting these terms on the right-hand side of the

$$(m+M)\ddot{X} + \frac{ml}{2}\ddot{\theta}\cos\theta - \frac{ml}{2}\sin\theta\dot{\theta}^2 + kX = Q_X = f_1(t) + f_2(t)\cos\theta$$

$$\frac{ml^2}{3}\ddot{\theta} + \frac{ml}{2}\ddot{X}\cos\theta - \frac{ml}{2}\sin\theta\dot{X}\dot{\theta} + \frac{ml}{2}\dot{X}\dot{\theta}\sin\theta + \frac{wl}{2}\sin\theta = Q_\theta = f_2(t)\, l \ . \tag{6.31}$$

differential Eqs.(6.29) gives:

Cart-Pendulum Example with Viscous Dissipation Forces

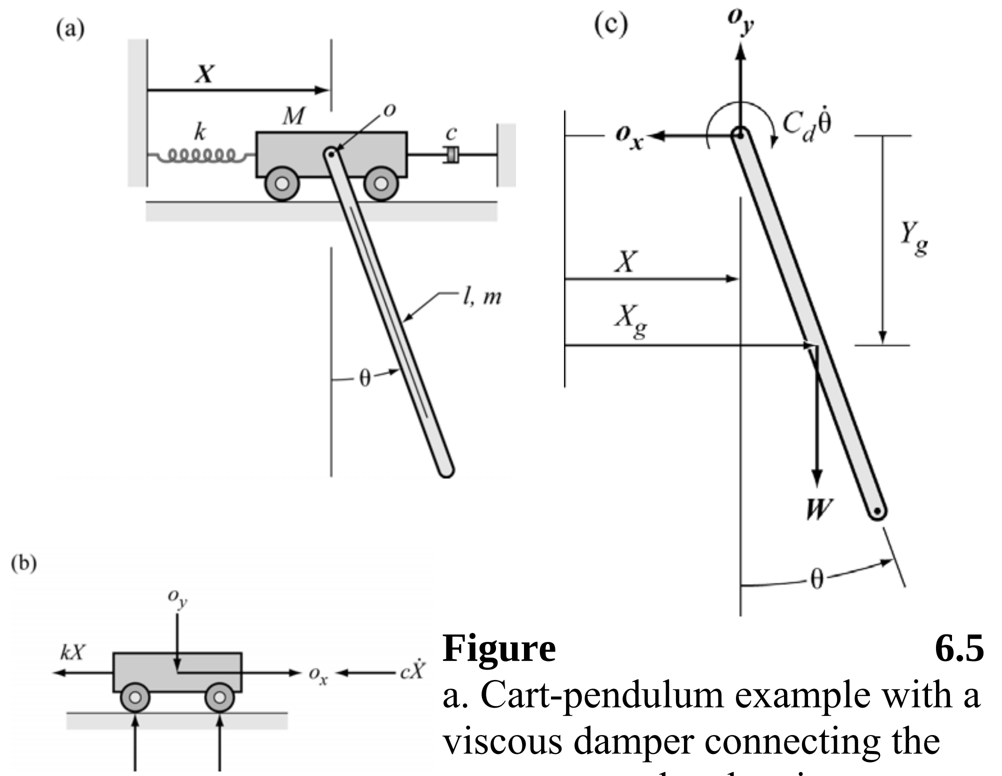

Figure 6.5 a. Cart-pendulum example with a viscous damper connecting the cart to ground and a viscous resistance moment at the pendulum support point *o*. b. Cart free-body diagram. c. Pendulum free-body diagram.

Figure 6.5a illustrates the cart-pendulum assembly with the addition of a viscous damper connecting the cart to ground. This damper is characterized by the linear damper coefficient *c*, yielding the damping force $\boldsymbol{f_{1d}} = -\boldsymbol{I} c\dot{X}$ acting on the cart. The point of application of the force is defined by the position vector $\boldsymbol{r_1} = \boldsymbol{I}X$. Also, suppose that we now have viscous damping in the pivot joint at *o*, supporting the pendulum. This type of energy dissipation moment was discussed in section 5.4, and is illustrated in figure 5.15. The resistance moment due to this damping is defined by $M_{o\theta} = -C_d\dot{\theta}$.

The following engineering-analysis task applies: *Use Lagrange's equations of motion to derive the equations of motion.* We have already developed the equations of motion for the system, except for the nonconservative generalized forces. To complete the assignment, we need to define the nonconservative forces due to damping and then plug them into the right-hand side of Eqs.(6.29).

Figure 6.5B-C present the free-body diagrams for the system, including this viscous force and moment. There is not enough space in the free-body diagram of figure 6.5B to show the viscous moment being reacted by the cart. The nonconservative differential work done by the dissipation

forces during a differential change in position of the system is

$$\begin{aligned} dWork_{n.c.} &= -C_d\dot{\theta}\,d\theta + \boldsymbol{f_{1d}}\bullet d\boldsymbol{r_1} = -C_d\dot{\theta}\,d\theta + (-\boldsymbol{I}c\dot{X}\bullet\boldsymbol{I}dX) \\ &= -C_d\dot{\theta}\,d\theta - c\dot{X}\,dX = Q_\theta\,d\theta + Q_X\,dX\ . \end{aligned} \tag{6.32}$$

Inserting Q_θ, Q_X on the right-hand side of Eqs.(6.29) gives the differential equation of motion:

$$\begin{aligned} &(m+M)\ddot{X} + \frac{ml}{2}\ddot{\theta}\cos\theta - \frac{ml}{2}\sin\theta\dot{\theta}^2 + kX = \\ &\qquad Q_X = -c\dot{X} \\ &\frac{ml^2}{3}\ddot{\theta} + \frac{ml}{2}\ddot{X}\cos\theta - \frac{ml}{2}\sin\theta\dot{X}\dot{\theta} + \frac{ml}{2}\dot{X}\dot{\theta}\sin\theta + \frac{wl}{2}\sin\theta = \\ &\qquad Q_\theta = -C_d\dot{\theta}\ . \end{aligned} \tag{6.33}$$

Again, the result is obtained more quickly and efficiently with Lagrange's equations than it could have been with a free-body/Newtonian approach.

Cart-Pendulum Example with a Coulomb-Friction Moment in the Pendulum Support Pivot.

Figure 6.6 provides a free-body diagram for the pendulum connection of the cart-pendulum example with Coulomb friction at the pivot-support point[1] o. Figure 6.6 shows the Coulomb friction force at the pivot pin due to the radial reaction force *N* and the Coulomb-friction coefficient μ. The Coulomb friction force is μN, opposing the rotation direction of the pendulum about the pivot pin. The pivot pin radius is e ; hence, the resistance moment of μN about the center of the pin is $-\mu e N sgn(\dot{\theta})$. In contrast to some of the preceding examples, no external forces act on either body, and Coulomb-friction at the joint is the only dissipation force. Stating the differential nonconservative work due to the Coulomb-friction moment would seem to be easy as

$$dWork_{n.c.} = -\mu e N sgn(\dot{\theta}) \cdot d\theta \ , \qquad (6.34)$$

which is simply the reaction moment times the differential rotation. The problem with this expression is that we don't

[1] This type of energy-dissipation moment was introduced in section 5.4, and is illustrated in figure 5.16.

know *N* and can only obtain it by drawing a free-body diagram and writing the components of $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{R}}_{\boldsymbol{g}}$ for the pendulum's mass center.

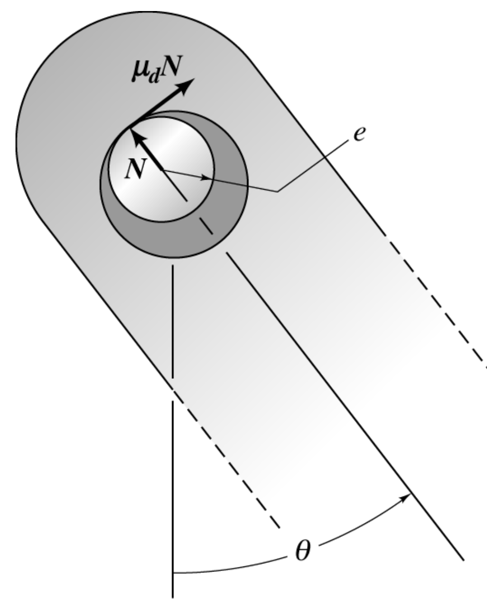

Figure 6.6 Free-body diagram for the cart-pendulum example with Coulomb friction acting at the pinned joint *o*.

From figure 6.5C:

$$\Sigma F_X = -o_X = m\ddot{X}_g \ , \quad \Sigma F_Y = o_Y - w = m\ddot{Y}_g \ . \qquad (6.35)$$

From figures 6.5C and 6.6, $N = o_Y \cos\theta + o_x \sin\theta$; however, to define the reaction components o_X, o_Y , we need to solve for $\ddot{X}_g, \ddot{Y}_g$ in terms of the generalized coordinates and their derivatives. From figure 6.5C

$$X_g = X + \frac{l}{2}\sin\theta \ , \quad Y_g = -\frac{l}{2}\cos\theta \ .$$

Differentiating these equations twice with respect to time gives

$$\ddot{X}_g = \ddot{X} + \frac{l}{2}\cos\theta\ddot{\theta} - \frac{l}{2}\sin\theta\dot{\theta}^2 \ , \quad \ddot{Y}_g = \frac{l}{2}\sin\theta\ddot{\theta} + \frac{l}{2}\cos\theta\dot{\theta}^2 \ .$$

Substituting $\ddot{X}_g, \ddot{Y}_g$ into Eqs.(6.35) to obtain o_X, o_Y , and then solving for N gives

$$N = W\sin\theta + M\frac{l}{2}\dot{\theta}^2 - M\ddot{X}\sin\theta \ .$$

Substituting into Eq.(6.34) gives

$$dWork_{n.c.} = -\mu e(W\sin\theta + M\frac{l}{2}\dot{\theta}^2 - M\ddot{X}\sin\theta)sgn(\dot{\theta})d\theta$$
$$= Q_\theta d\theta + Q_X dX \ .$$

Hence,

$$Q_\theta = -\mu e(W\sin\theta + M\frac{l}{2}\dot{\theta}^2 - M\ddot{X}\sin\theta)sgn(\dot{\theta}) \ , \ Q_X = 0 \ .$$

Substituting these results back into Eq.(6.29) completes the task as

$$(m+M)\ddot{X} + \frac{ml}{2}\ddot{\theta}\cos\theta - \frac{ml}{2}\sin\theta\dot{\theta}^2 + kX = Q_X = 0 \ ,$$

$$\frac{ml^2}{3}\ddot{\theta} + \frac{ml}{2}\ddot{X}\cos\theta - \frac{ml}{2}\sin\theta\dot{X}\dot{\theta} + \frac{ml}{2}\dot{X}\dot{\theta}\sin\theta + \frac{wl}{2}\sin\theta = Q_\theta$$
$$= -\mu e(W\sin\theta + M\frac{l}{2}\dot{\theta}^2 - M\ddot{X}\sin\theta)sgn(\dot{\theta}).$$

For this example, the perceived advantages of the Lagrangian approach over a free-body/Newtonian approach largely vanishes because: (i) a free-body diagram is required, and (ii) acceleration terms are needed for the pendulum's mass center.