P2 FA10 - Derive EOM for simple mechanical system

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For the system shown in the figure, $Z_{(t)}=Z_o \cos(\omega t)$ is a periodic displacement input (known). Perform the following tasks:

a) Set $X(t)$ as the coordinate for the block motion; $X=0$ denotes the unstretched length of spring 1. Draw a free body diagram and derive the block equation of motion. You should get: $M\ddot{X} + C\dot{X} + (K_1+K_2)X_{(t)} = K_2 Z_{(t)}$ (Must demonstrate how you derive the equation)

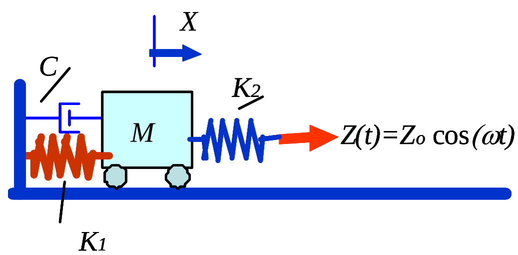
b) For K_1=250 N/m, K_2=400 N/m, M=50 gram and C=0.75 N.s/m, find the system natural frequency (Hz) and damping ratio (ζ).

c) For Z_o=15 mm, ω=20 Hz, determine the periodic forced response of the block X(t), i.e., calculate the amplitude of motion and the phase lag relative to Z(t). Please explain where operation is above or below or around the natural frequency; most importantly, is operation safe?

d) BONUS: Determine the excitation frequency range where the amplitude of motion $|X_{max}|$=30 mm is not exceeded.

$M := 0.05 \cdot kg \qquad K_1 := 250 \cdot \frac{N}{m}$

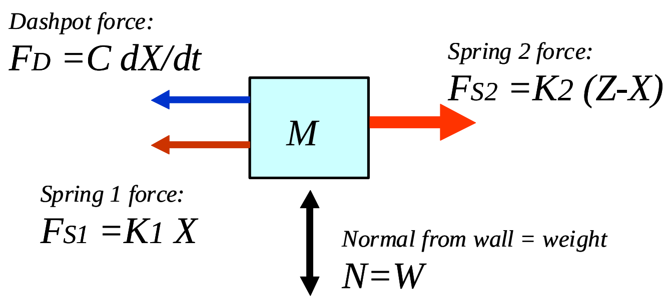
$C := 0.75 \cdot \frac{N \cdot s}{m} \qquad K_2 := 400 \cdot \frac{N}{m}$



(a.1) FBD and forces

For FBD diagram, assume a state of motion such that $Z > X > 0$

X=0 means unstretched position of spring 1



(1)

(a.2) derive EOM

From the FBD diagram, Newton's 2nd law states:

$$M \cdot \frac{d^2}{dt^2}X = F_{S2} - F_{S1} - F_D \qquad \textbf{(2)}$$

where $F_{Damper} = C \cdot \frac{d}{dt}x$ (3) is the dashpot force

$N = W$ wall reaction force

$F_{S1} = K \cdot X$

$F_{S2} = K_2 \cdot (Z - X)$

(4) Elastic forces from springs 1 and 2. Spring 2 drives motion of block.

Substitute Eqs. (3,4) into Eq. (2) to get

$$M \cdot \frac{d^2}{dt^2}X + C \cdot \frac{d}{dt}X + K_{eq} \cdot X = K_2 \cdot Z(t) \qquad \textbf{(5)}$$

where

$Z(t) = Z_o \cdot \cos(\omega \cdot t)$

and

$K_{eq} := K_1 + K_2$

(b) Calculate natural frequency (fn) and viscous damping ratio (ζ):

$$\omega_n := \left(\frac{K_{eq}}{M}\right)^{.5} \qquad \omega_n = 114.018 \cdot \frac{1}{sec} \qquad f_n := \frac{\omega_n}{2 \cdot \pi}$$

$$f_n = 18.146 \cdot Hz$$

$$\zeta := \frac{C}{2 \cdot (K_{eq} \cdot M)^{.5}} \qquad \zeta = 0.066$$

(6)

(c) Calculate response for

$$Z_o := 0.015 \cdot m \quad f := 20 \cdot Hz \qquad \omega := f \cdot 2 \cdot \pi$$

$$Z(t) := Z_o \cdot \cos(\omega \cdot t)$$

The system response will be $X(t) := X_{op} \cdot \cos(\omega \cdot t + \varphi)$ **(7)**

Where X_{op} and φ are the amplitude and phase lag of the motion X(t)

Find frequency ratio $r := \frac{\omega}{\omega_n}$ $r = 1.102$ operation just above natural frequency

Using FRF formula:

Find Amplification factor

$$A(r) := \frac{1}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}}$$

$$A(r) = 3.86$$

Find **Amplitude at steady-state**

note here that $F_o = K_2 Z_o$

$$X_{op} := \frac{Z_o \cdot K_2}{K_{eq}} \cdot A(r) \qquad X_{op} = 0.036 \cdot m$$

Find **phase lag**

$$\varphi := \operatorname{atan}\left(\frac{-2 \cdot \zeta \cdot r}{1 - r^2}\right) - \pi$$

since r>1

$$\frac{X_{op}}{Z_o} = 2.375$$

large amplitude ratio - NOT very safe

$$\varphi = -2.548 \quad rad$$

$$\varphi \cdot \frac{180}{\pi} = -145.97 \quad degrees$$

(d) Frequency range for max amplitude

Let $X_{max} := 30 \cdot mm$

from

$$X_{op} := Z_o \cdot \frac{K_2}{K_{eq}} \cdot A(r)$$

Let

$$\delta := \frac{X_{max}}{Z_o} \cdot \frac{K_{eq}}{K_2}$$

then

$$\delta^2 = A(r)^2 = \frac{1}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]}$$

$$\delta = 3.25$$

expanding:

$$\delta^2 \cdot \left[\left(1 - 2 \cdot s + s^2\right) + (2 \cdot \zeta)^2 \cdot s \right] = 1 \qquad \text{where} \quad s = r^2$$

collecting like terms $\quad \delta^2 \cdot s^2 + \left(4 \cdot \zeta^2 - 2 \cdot \delta^2\right) \cdot s + \left(\delta^2 - 1\right) = 0$

and identifying this polinomial with $\quad a \cdot s^2 + b \cdot s + c = 0$

$$a := \delta^2 \qquad b := \left(4 \cdot \zeta^2 - 2 \cdot \delta^2\right) \qquad c := \left(\delta^2 - 1\right)$$

solve for the two roots

$$s_1 := \frac{-b - \left(b^2 - 4 \cdot a \cdot c\right)^{.5}}{2 \cdot a} \qquad s_1 = 0.694$$

$$s_2 := \frac{-b + \left(b^2 - 4 \cdot a \cdot c\right)^{.5}}{2 \cdot a} \qquad s_2 = 1.304$$

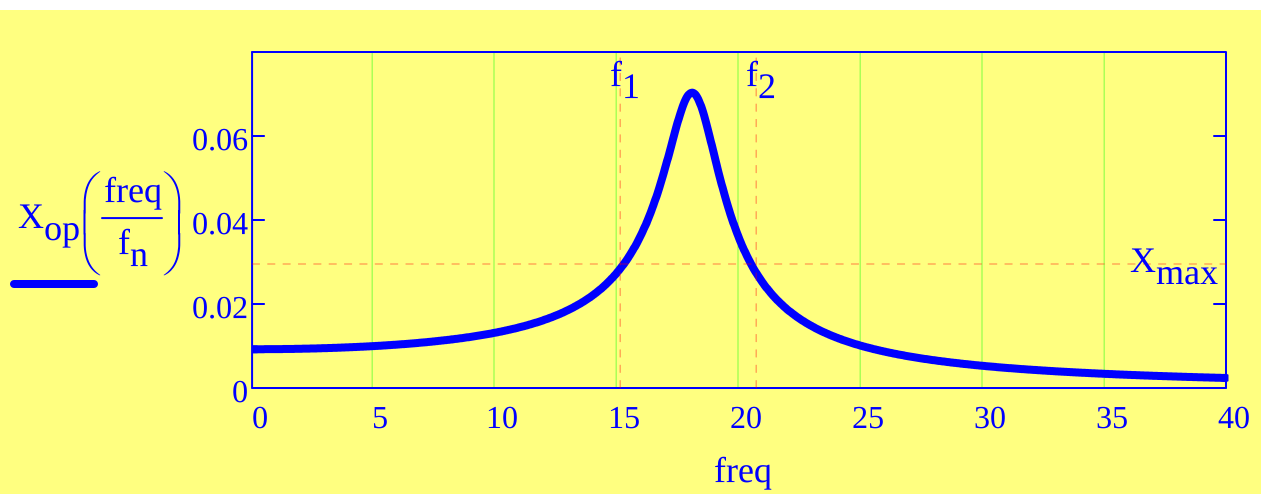
hence, the two frequencies below and above which the amplitude ratio will be less than Xmax are:

$$f_1 := s_1^{0.5} \cdot f_n \qquad f_2 := s_2^{0.5} \cdot f_n$$

$f_n = 18.146 \cdot Hz \qquad f_1 = 15.119 \cdot Hz \qquad f_2 = 20.724 \cdot Hz$

For comparison purposes, lets plot the function

$$X_{op}(r\_) := Z_o \cdot \frac{K_2}{K_{eq}} \cdot \frac{1}{\left[\left(1 - r\_^2\right)^2 + (2 \cdot \zeta \cdot r\_)^2\right]^{.5}}$$

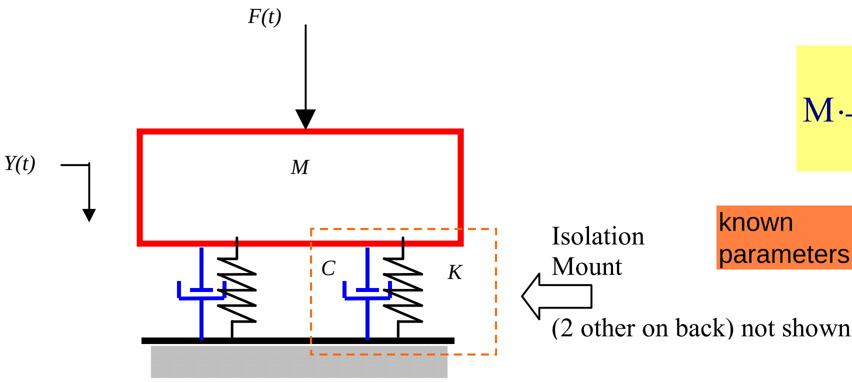


MEEN 363 - SP09 - Exam 2 - Problem 1 (Periodic Forced Response)

A machine weighing W=2200 lb is known to cause unacceptable vibrations in nearby equipment. The engineering modification consists of mounting the machine on **four vibration isolators,** as shown schematically in the figure. Upon installation, the **static deflection** of each isolator is 1 inch. Each isolator has a viscous damping coefficient equal to C=2.8 lb.s/in.

There is a periodic load excitation acting **on** the machine, i.e. $F=F_o \cos(\omega t)$, where $F_o=1000$ lb and frequency $f=(\omega/2\pi)=7$ Hz, determine

a) System natural frequency (f_n [Hz]) and damping ratio (ξ). [10]
b) Amplitude [inch] and phase lag [degrees] of system motion $Y(t)$ at the excitation frequency (f=7 Hz). [10]
c) Sketch the response $Y(t)$ versus time, label physical dimensions in graph. [5]
d) The amplitude of motion if (**for some unfortunate reason**) a periodic force with a frequency equal to the system natural frequency, i.e. $F=F_o \sin(\omega_n t)$, is exerted **on** the machine. [5]



EOM with Y origin from SEP:

$$M \cdot \frac{d^2}{dt^2} Y + K \cdot Y + C \cdot \frac{d}{dt} Y = F_o \cdot \cos(\omega \cdot t)$$

known parameters

$M := 2200 \cdot lb$

$C_i := 2.8 \cdot lbf \cdot \frac{s}{in}$ $\qquad g = 386.089 \frac{in}{s^2}$

static deflection: $\delta := 1 \cdot in$ $\qquad K_i := \frac{M \cdot g}{4 \cdot \delta}$ $\qquad K_i = 550 \frac{lbf}{in}$ isolator stiffness

Each isolator supports 1/4 the machine weight.

Total stiffness: $K := 4 \cdot K_i$ $\qquad C := 4 \cdot C_i$

$K = 3.853 \times 10^5 \frac{N}{m}$

Force magnitude: $F_o := 1000 \cdot lbf$

$C = 1.961 \times 10^3 N \cdot \frac{s}{m}$

frequency of forcing function: $\omega := 7 \cdot 2 \cdot \pi \cdot \frac{rad}{s}$

a) Natural frequency and damping ratio:

$\omega_n := \left(\frac{K}{M}\right)^{.5}$ $\qquad \zeta := \frac{C}{2 \cdot (K \cdot M)^{.5}}$ $\qquad \omega_n = 19.649 \frac{rad}{s}$ $\qquad f_n := \frac{\omega_n}{2 \cdot \pi}$ $\qquad f_n = 3.127 Hz$

$\zeta = 0.05$

static deflection: $Y_s := \frac{F_o}{K}$ $\qquad Y_s = 0.455 in$

b) for harmonic excitation:

$F(t) := F_o \cdot \cos(\omega \cdot t)$

compare to response if M=C=0 (low frequency)

The machine periodic response is $Y(t) = Y_{op} \cdot \cos(\omega \cdot t + \phi)$ $\qquad Y_S(t) := \frac{F_o}{K} \cdot \cos(\omega \cdot t)$

$$Y_{op} = Y_s \cdot H(r)$$

Find frequency ratio $r := \frac{\omega}{\omega_n}$

$r = 2.238$ Operation above natural frequency

The amplification factor and phase angle are:

$$H(r) := \frac{1}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \qquad \phi(r) := \operatorname{atan}\left(\frac{-2 \cdot \zeta \cdot r}{1 - r^2}\right) - \pi$$

since r >1

$H(r) = 0.249$ $\phi(r) = -3.086$ $\phi(r) \cdot \frac{180}{\pi} = -176.804$ degrees

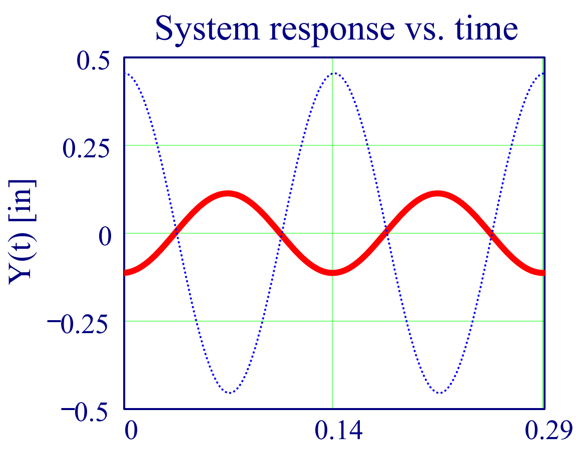
$Y_{op} := Y_s \cdot H(r)$ $Y_{op} = 0.113\,in$ **<< Ys** $Y_s = 0.455\,in$

(c) graph of periodic response $Y(t) := Y_{op} \cdot \cos(\omega \cdot t + \phi(r))$ $T := \frac{2 \cdot \pi}{\omega}$

for two periods of motion

period

$T = 0.143\,s$



Note the response at t=0 shows a negative displacement. This is since the phase angle is close to -180 degrees.

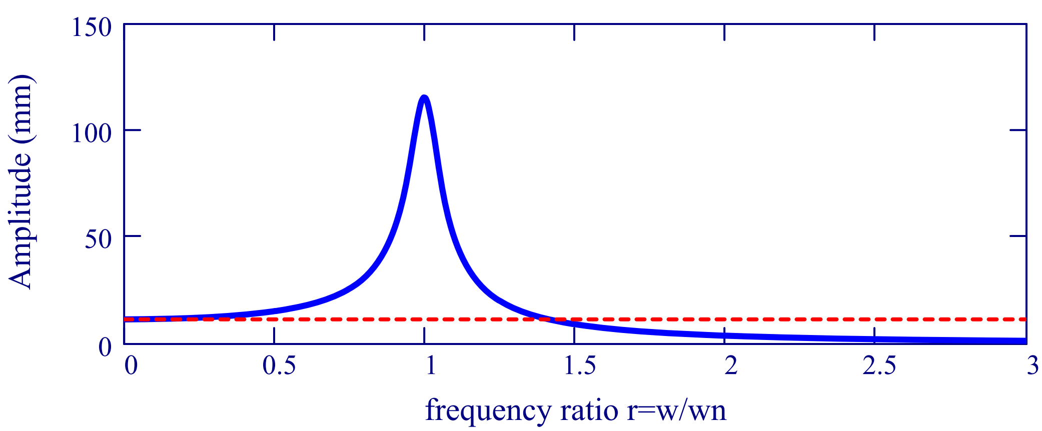
Broken-line shows curve of F(t)/K

(d) for excitation at the natural frequency: $r := 1$ $H(r) = 9.997$

$Y_{opn} := Y_s \cdot H(r)$

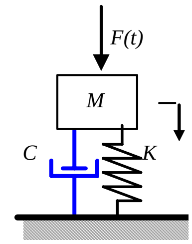
$\frac{Y_s}{2 \cdot \zeta} = 4.544\,in$ $Y_{opn} = 4.544\,in$ Phase lag is $-\pi/2$ rads (-90 degrees)

(e) Isolators are effective at frequency of operation (r>1.44), where $Y_{op} < Y_s$



MEEN 363 FALL 06 EXAM 2 PROBLEM 2

The figure shows a simple mechanical system (*K-C-M*) excited by a harmonic force $F(t)=F_o \sin(\omega t)$, F_o=50 lb_f. The frequency of the external force excitation ranges from 4 Hz to 20 Hz. Tests show the natural frequency is f_n (10 Hz) and the viscous damping coefficient (*C*) equals 4 lb_f.s/in. The mass of the system is *M*=120 lb.



a) Determine the system steady-state **amplitude** of motion (in inch) and **phase lag** (in degrees) for excitation at 15 Hz. [10]

b) Repeat (a) with **stiffness equal to twice the original value**. By how much the amplitude of motion is reduced or amplified? [6]

c) Repeat (a) with a **mass equal to twice the original value**. By how much the amplitude of motion is reduced or amplified? [6]

d) Based on the results (a-c) what system configuration will you select? Are there other options? Explain. [3]

DATA: $F(t) = F_o \cdot \sin(\omega \cdot t)$ $f_n := 10 \cdot Hz$ $F_o := 50 \cdot lbf$ frequency of operation between 4-20 Hz

KEY: $M := 120 \cdot lb$ $C := 4 \cdot lbf \cdot \frac{sec}{in}$ $W := M \cdot g$

$\omega_n := f_n \cdot 2 \cdot \pi$ $\omega_n = 62.832 \frac{rad}{sec}$

Analyze excitation at

$f := 15 \cdot Hz$ $\omega := f \cdot 2 \cdot \pi$

First determine the ORIGINAL system stiffness (K) and damping ratio (ζ):

$K := \omega_n^2 \cdot M$ $K = 1.227 \times 10^3 \frac{lbf}{in}$

$\zeta := \frac{C}{2 \cdot M \cdot \omega_n}$ $\zeta = 0.102$

Let: $K_{orig} := K$

$M_{orig} := M$

$\zeta_{orig} := \zeta$ for later use

Basic Knowledge as stated on Summary of formulas:

The system periodic response is $Y(t) = \delta_s \cdot H \cdot \sin(\omega \cdot t + \Psi)$ where $\delta_s = \frac{F_o}{K}$ is the static displacement,

H is the amplification factor and Ψ the phase angle defined as a function of the frequency ratio (r=ω/ωn):

$$H(r,\zeta) := \frac{1}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \qquad \Psi(r,\zeta) := \operatorname{atan}\left(\frac{-2 \cdot \zeta \cdot r}{1 - r^2}\right) \quad \text{with} \quad r = \frac{\omega}{\omega_n}$$

(a) ORIGINAL system $r := \frac{f}{f_n}$ $r = 1.5$ operation above natural frequency since r>1

$Y_{op\_original} := \frac{F_o}{K} \cdot H(r,\zeta)$ **amplitude:** $Y_{op\_original} = 0.032\ in$ **phase lag:** $\Psi(r,\zeta) \cdot \frac{180}{\pi} - 180 = -166.191$ degrees

(b) If the stiffness is doubled: $K := 2 \cdot K_{orig}$

Natural frequency increases and damping ratio decreases respect to original system

NEW: $\omega_n := \left(\frac{K}{M}\right)^{.5}$ $\zeta := \frac{C}{2 \cdot M \cdot \omega_n}$ $\omega_n = 88.858 \frac{rad}{sec}$ $\zeta = 0.072$ $r := \frac{\omega}{\omega_n}$ $r = 1.061$ >1

$Y_{op\_b} := \frac{F_o}{K} \cdot H(r,\zeta)$ **amplitude:** $Y_{op\_b} = 0.103\ in$ **phase lag:** $\Psi(r,\zeta) \cdot \frac{180}{\pi} - 180 = -129.135$ degrees

$\frac{Y_{op\_b}}{Y_{op\_original}} = 3.25$ **amplitude increases 3.25 times, because operation is too close to resonance. Furthermore, damping ratio is smaller than original.**

(c) If the mass is doubled: $K := K_{orig}$ $M := 2 \cdot M_{orig}$

Natural frequency decreases and damping ratio decreases respect to original system

NEW: $\omega_n := \left(\frac{K}{M}\right)^{.5}$ $\zeta := \frac{C}{2 \cdot M \cdot \omega_n}$ $\omega_n = 44.429 \frac{rad}{sec}$ $\zeta = 0.036$ $r := \frac{\omega}{\omega_n}$ $r = 2.121$ >1

$Y_{op\_c} := \frac{F_o}{K} \cdot H(r, \zeta)$ **amplitude:** $Y_{op\_c} = 0.012\ in$ **phase lag:** $\Psi(r,\zeta) \cdot \frac{180}{\pi} - 180 = -177.487$ degrees

nearly - 180 deg, since r >> 1

$\frac{Y_{op\_c}}{Y_{op\_original}} = 0.367$ **Amplitude decreases to 0.37 times, because operation is well above natural frequency.**

(d) Select case (c), i.e. double mass since it shows the lowest amplitude of response. The system effectively works as an isolator. Doubling the stiffness is certainly NOT a good choice since natural frequency is too close to excitation frequency.

Another option?

Keep original system and DOUBLE damping? Not really, see below:

$\zeta_{orig} = 0.102$

$f_n = 10\,Hz$

$f = 15\,Hz$

$Y_{op} := \frac{F_o}{K_{orig}} \cdot H\left(\frac{f}{f_n}, 2 \cdot \zeta_{orig}\right)$ r= $\frac{f}{f_n} = 1.5$

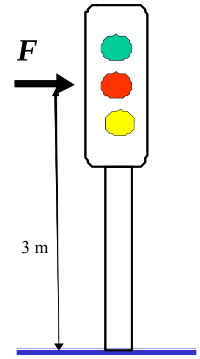
$Y_{op} = 0.029\ in$ compared to: $Y_{op\_original} = 0.032\ in$

very little reduction. Why? already operating above natural frequency. Damping does little to reduce vibration amplitude!

P2. Periodic forced response of a SDOF mechanical system. DESIGN COMPONENT

The signal lights for a rail may be modeled as a 176 lb mass mounted 3 m above the ground of an elastic post. The natural frequency of the system is measured to be 12.2 Hz. Wind buffet generates a horizontal harmonic force at 12 Hz. The light filaments will break if their peak accelerations exceed *15g*. Determine the maximum acceptable force amplitude $|F|$ when the damping ratio ζ=0.0 and 0.01.

Full grade requires you to explain the solution procedure with due attention to physical details



The excitation force is periodic, say F(t)=Fo sin(ωt). then the system response will also be periodic, Y(t), with same frequency as excitation. Assuming steady state conditions:

STEADY RESPONSE of M-K-C system to PERIODIC Force with frequency ω

Case: periodic force of constant magnitude Define operating frequency ratio: $r = \frac{\omega}{\omega_n}$

$$F(t) = F_o \cdot \sin(\omega \cdot t)$$

System periodic response: $Y(t) = \delta s \cdot H(r) \cdot \sin(\omega \cdot t + \Psi)$ (1)

where:

$$\delta_s = \frac{F_o}{K_e} \qquad H(r) = \frac{1}{\left[\left(1-r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \qquad \tan(\Psi) = \frac{-2 \cdot \zeta \cdot r}{1-r^2}$$

care with angle, range: 0 to -180deg

From (1), the acceleration is $a(t) = -\omega^2 \cdot Y(t) = A \cdot \sin(\omega(t + \Psi - 180))$

the magnitude of acceleration is $A = \frac{F_o}{K_e} \cdot \frac{\omega^2}{\left[\left(1-r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}}$ or $A = \frac{F_o}{M_e} \cdot \frac{r^2}{\left[\left(1-r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}}$

hence, define $A_{max} := 10 \cdot g$ maximum allowed acceleration of filament

$HZ := 2 \cdot \pi \cdot \frac{1}{s}$

system mass $M_e := 150 \cdot lb$

$f_n := 12 \cdot HZ$ natural frequency

$f := 11.5 \cdot HZ$ excitation frequency due to wind buffets

Let $r_o := \frac{f}{f_n}$ $\quad r_o = 0.958$ close to natural frequency

The maximum force allowed equals

$$F_{max}(r, \zeta) := A_{max} \cdot M_e \cdot \frac{\left[\left(1-r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}}{r^2}$$

without any damping $F_{max}(r_o, 0) = 133.27\,\text{lbf}$ Note the importance of damping that leads to a substantial increase in force allowed

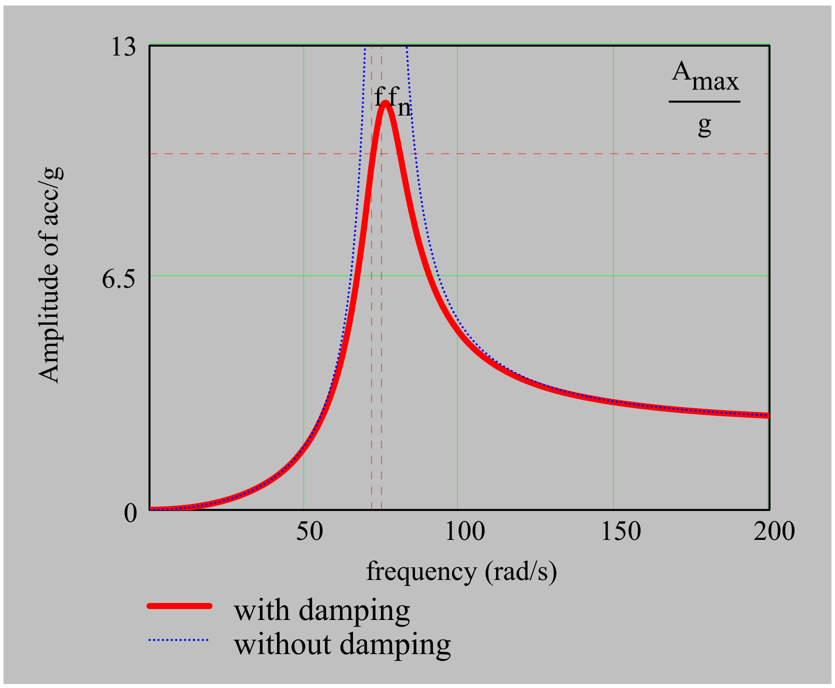
with damping $\xi := 0.1$ $F_{max}(r_o, \xi) = 340.231\,\text{lbf}$

$$\frac{F_{max}(r_o, \xi)}{F_{max}(r_o, 0)} = 2.553$$

For the force found the amplitude of acceleration is

$$F_o := F_{max}(r_o, \xi)$$

$$A(r, \zeta) := \frac{F_o}{M_e} \cdot \frac{r^2}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}}$$



$f_n = 75.398\,\frac{1}{s}$

$\frac{1}{2 \cdot \xi} = 5$

GRAPH NOT FOR EXAM

SInce $r_o \sim 1$, a simpler enginering formula gives $A_{max} \cdot M_e \cdot 2 \cdot \xi = 300\,\text{lbf}$

which gives a very good estimation of the maximum wind force allowed

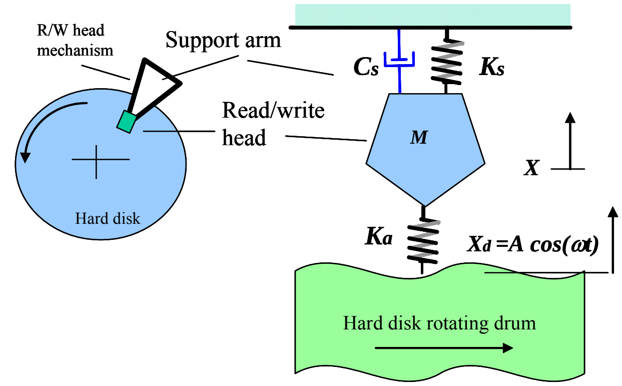
b) a system with damping $\xi = 0.1$ will produce a 255 % increase in allowable force
Hence, the rail lightsystem will be more reliable, lasting longer.

$$\frac{\ldots}{F_{max}(r_o, 0)} = 2.553$$

c) Posts are usually hollow for the cables to be routed. These posts have layers of elastomeric material (~rubber-like) inside to increase their structural damping. Modern posts are wound up fro composites that integrate damping layers. Clearly, adding a "true" dashpot is not cost-effective

MEEN 363/617 Example Base motion - Frequeny response

The figure displays a schematic view of a read/write head in a hard disk. The support arm holding the R/W head is represented by structural stiffness $K_s=10^5$N/m, and damping coefficient C_s [Ns/m] to be determined. The R/W head mass equals 25 gram. $K_a=10^6$N/m represents the stiffness of the air film between the rotating disk and the R/W head. When the disk spins, micro-asperities in the disk induce a wavy-like motion represented as $X_d=A\cos(\omega t)$, where A= 1 micro-meter. The frequency ($f=\omega/2\pi$) of the excitation varies from 500 Hz to 4000 Hz, depending on the rotational speed of the disk and the radial position of the R/W head support structure.
The origin of the coordinate systems, $X=X_d=0$, represents the static equilibrium position.



a) Draw a free body diagram, derive the equation of motion for the R/W head system, and calculate the system natural frequency (Hz) [5, 5, 5]
b) State a formula for the motion X(t), i.e. amplitude and phase angle as function of frequency [5]
c) When $\omega=\omega_n$, find the magnitude of the damping coefficient C_s such that the amplitude of motion for the R/W hear does not exceed 2 micro-meter. [5]
d) **Sketch** the amplitude and phase for the frequency response of the R/W head. Your graphs must show physical dimensions and explanatory sentences explaining relevant information related to the system motion [2 x 5]

DATA: $K_a := 10^6 \cdot \frac{N}{m}$ $\qquad K_s := \frac{K_a}{10}$

$M := 0.025 \cdot kg$ $\qquad \zeta$: TBD

KEY:

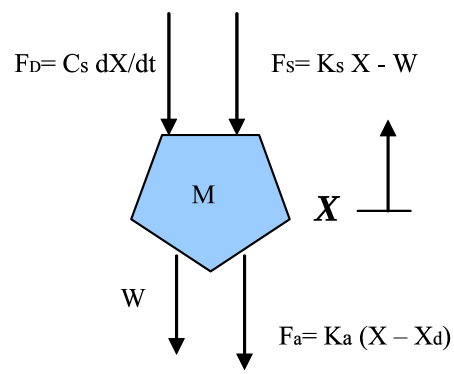
Assume X >Xd

(a) Draw FBD, derive EOM, find ω_n

Define coordinate systems with origin at static equilibrium position.

Upon assembly, the arm support spring holds the weight of the R/W head.

Note that in actuality, the air film stiffness is zero when the disk does not spin.



Apply Newtons law

$$M \cdot \frac{d^2}{dt^2}X = -F_D - F_s - F_a - W$$

The Equation of motion for the system is

$$M \cdot \frac{d^2}{dt^2}X + K_e \cdot X + C_s \cdot \frac{d}{dt}X = K_a \cdot X_d = F(t)$$

where $K_e := K_a + K_s$ $\qquad$ for periodic motions $X_d = A \cdot \cos(\omega \cdot t)$ then $F(t) = A \cdot K_a \cdot \cos(\omega \cdot t)$

$A := 1 \cdot 10^{-6} \cdot m$

natural frequency

$\omega_n := \left(\frac{K_e}{M}\right)^{0.5}$ $\qquad f_n := \frac{\omega_n}{2 \cdot \pi}$ $\qquad f_n = 1.056 \times 10^3\,Hz$ $\qquad \omega_n = 6.633 \times 10^3 \frac{rad}{s}$

(b) Establish system periodic response

The system periodic response is: (from cheat sheet)

$$X(t) = \delta \cdot H \cdot \sin(\omega \cdot t + \Psi) = X_A \cdot \sin(\omega \cdot t + \Psi) \qquad X_A = \delta \cdot H(r)$$

where $\delta := \dfrac{A \cdot K_a}{K_e}$ is the "static" displacement $\delta = 9.091 \times 10^{-7}\, m$

H is the amplification factor and Ψ is the phase angle defined as a function of the frequency ratio (r=ω/ωn):

$$H(r) = \frac{1}{\left[\left(1-r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \qquad \tan(\Psi) = \frac{-2 \cdot \zeta \cdot r}{1 - r^2} \qquad \text{with} \quad r = \frac{\omega}{\omega_n}$$

(c) Calculate damping coefficient needed for amplitude of motion NOT to exceed 2 microns

$X_{Amax} := 2 \cdot 10^{-6} \cdot m$

Define ratio $\eta := \dfrac{X_{Amax}}{\delta}$ $\eta = 2.2$

and from equation for amplification factor H: $$\eta^2 = \frac{1}{\left[\left(1-r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]}$$

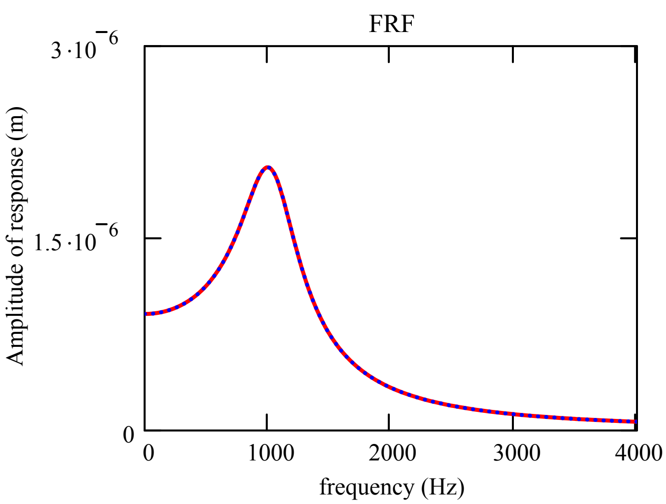
for operation at the natural frequency r=1 $\eta = \dfrac{1}{2 \cdot \zeta}$ Hence $\zeta := \dfrac{1}{2 \cdot \eta}$

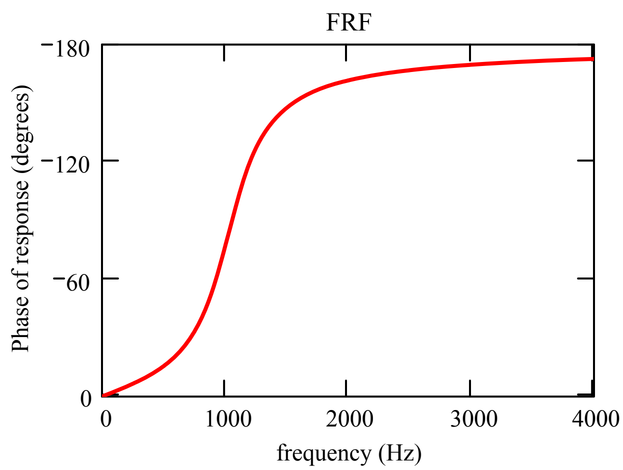
$\zeta = 0.227$

$C_s := \zeta \cdot 2 \cdot (K_e \cdot M)^{0.5}$ $C_s = 75.378\, N \cdot \dfrac{s}{m}$ Value of damping coefficient

Functions for graphs

(d) FRF graphs (amplitude and phase) of R/W head versus frequency range of interest





Note that at low frequencies, amplitude equals $\delta = 9.091 \times 10^{-7}\,m$ and phase lag is 0 degrees

at high frequencies, amplitude approaches null values and phase lag approaches 180 degrees

at the natural frequency, amplitude of motion is $X_{Amax} = 2 \times 10^{-6}\,m$

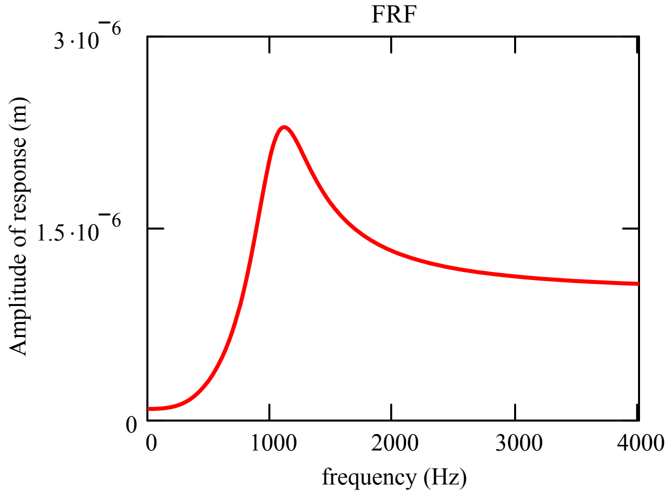
NOT for exam:

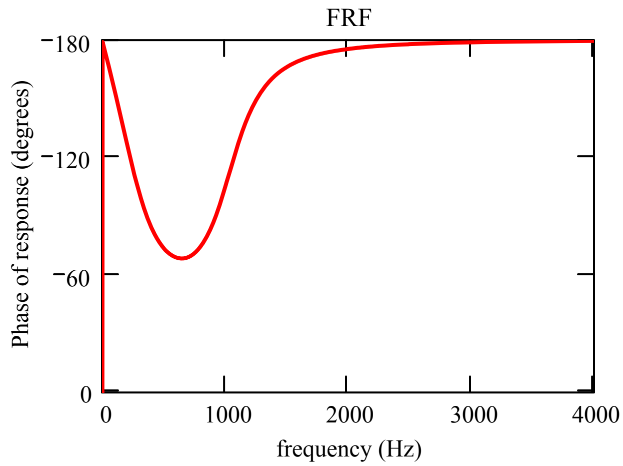
Relative motion is of importance, i.e. Y=X-Xd

$$Y(\omega) := X(\omega) - A$$

$$Y_A(\omega) := |Y(\omega)|$$

$$\Psi_Y(\omega) := \frac{180}{\pi} \cdot \arg(Y(\omega))$$





$$Y(\omega_n \cdot 0) = -9.091 \times 10^{-8}\,m$$

$$\delta - A = -9.091 \times 10^{-8}\,m$$

Phase angle cannot be positive

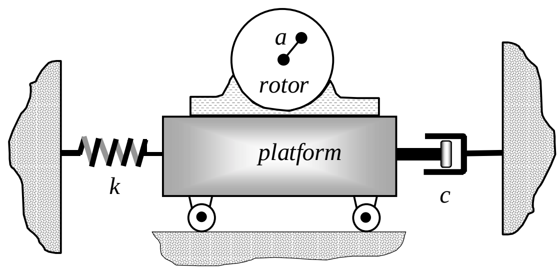
MEEN 363 - QUIZ 3 Names:

The rotor of an electric generator weights 750 lb and is attached to a platform weighing 7750 lb. The motor has an imbalance eccentricity (a) of 2.5 mils. The platform can be modeled as shown in the figure. The equivalent stiffness (K) of the platform is 1 million-lb/in, and the equivalent damping is C=100 lb-s/in. The operating speed of the generator is 1800 rpm.

(a) determine the response of the platform (amplitude and phase) at the operating speed.

(b) determine the response of the platform (amplitude and phase) if the rotor spins with a speed coinciding with the system natural frequency.

(c) if the platform stiffness is increased by 25%, determine the allowable amount of imbalance (a) that will give the same amplitude of motion as determined in (a). Assume that the mass of the platform and the damping do not change appreciably by performing the stiffening.



KEY: System excitation due to rotating imbalance

Given $K := 10^6 \cdot \frac{lbf}{in}$ $M_{rotor} := 750 \cdot lb$ $M_{platform} := 7750 \cdot lb$

$M := M_{rotor} + M_{platform}$ $C := 100 \cdot lbf \cdot \frac{sec}{in}$

$M = 8.5 \times 10^3\, lb$

calculate the system natural frequency and damping ratio:

$\omega_n := \left(\frac{K}{M}\right)^{.5}$ $\zeta := \frac{C}{2 \cdot M \cdot \omega_n}$

$\omega_n = 213.125 \frac{rad}{s}$ $\zeta = 0.011$ little damping

and operating frequency ratio (r) for rotor speed: $RPM := 1800$

$\omega := RPM \cdot \frac{2 \cdot \pi}{60} \cdot \frac{rad}{s}$ $\omega = 188.496 \frac{rad}{s}$ $r := \frac{\omega}{\omega_n}$ $r = 0.884$

The system response (amplitude and phase) for imbalance excitation are: $a := 2.5 \cdot 10^{-3} \cdot in$

$$Y_{op}(r) := a \cdot \frac{M_{rotor}}{M} \cdot \frac{r^2}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}}$$

$$\Psi(r) := atan\left(\frac{-2 \cdot \zeta \cdot r}{1 - r^2}\right)$$

Thus, at $r = 0.884$ <1

$Y_{op}(r) = 7.894 \times 10^{-4}\,in$ $\Psi(r)\cdot\frac{180}{\pi} = -4.947$ [degrees]

Let: $Y_{oper} := Y_{op}(r)$

(b) If the rotor should spin with a speed coinciding with the system natural frequency,

$r := 1$

$\omega := \omega_n$

$RPM := \omega\cdot\frac{60}{2\cdot\pi}\cdot\frac{s}{rad}$ $RPM = 2.035 \times 10^3$

the system response is

$Y_{op}(1) = 0.01\,in$ $\Psi := -90$ degrees $\frac{Y_{op}(1)}{Y_{oper}} = 13.111$

also determined from:

$$a\cdot\frac{M_{rotor}}{M}\cdot\frac{1}{2\cdot\zeta} = 0.01\,in$$

(c) If the platform increases K by 25%, $K_{original} := K$

$K := 1.25\cdot K$ to maintain $Y_{oper} = 7.894 \times 10^{-4}\,in$

calculate the NEW system natural frequency and damping ratio:

$$\omega_n := \left(\frac{K}{M}\right)^{.5} \qquad \zeta := \frac{C}{2\cdot M\cdot\omega_n}$$

$\omega_n = 238.281\,\frac{rad}{s}$ $\zeta = 9.531 \times 10^{-3}$ small change in damping ratio

and operating frequency ratio (r) for rotor speed: $RPM := 1800$

$\omega := RPM\cdot\frac{2\cdot\pi}{60}\cdot\frac{rad}{s}$ $\omega = 188.496\,\frac{rad}{s}$ $r := \frac{\omega}{\omega_n}$ $r = 0.791$

from relationship:

$$Y_{op}(r) := a\cdot\frac{M_{rotor}}{M}\cdot\frac{r^2}{\left[\left(1-r^2\right)^2 + (2\cdot\zeta\cdot r)^2\right]^{.5}}$$

determine the (new) allowable imbalance:

$$a := \frac{Y_{oper}}{\left[\frac{M_{rotor}}{M}\cdot\frac{r^2}{\left[\left(1-r^2\right)^2 + (2\cdot\zeta\cdot r)^2\right]^{.5}}\right]}$$

$a = 5.354 \times 10^{-3}\,in$

i.e. ~ twice as original imbalance
(eccentricity) displacement,

Example: Base motion excitation

An instrumentation sensor with mass 0.5 lb is attached to the casing of a steam turbine running at 3600 rpm. The casing motion has an amplitude of 0.005 in. Test show that the mass supported by the sensor attached has a natural frequency of 70 Hz. How much damping (lb.s/in) is needed to keep the sensor steady state amplitude below 0.015 in? Calculate the sensor stiffness (lb/in).

KEY:

This is a typical problem in which the source of excitation for the sensing element is **base motion**. The sensing element is a simple spring-mass-damper system.

a) calculate frequency ratio (r) giving us information on the regime of operation (below, around or above the natural frequency)

excitation frequency: $RPM := 3600$ $\qquad W := 0.5 \cdot lbf \qquad M := 0.5 \cdot lb$

$$\omega := RPM \cdot \frac{2 \cdot \pi}{60} \cdot \frac{rad}{s} \qquad \omega = 376.99 \frac{rad}{s}$$

natural frequency:

$$\omega_n := 70 \cdot 2 \cdot \pi \cdot \frac{rad}{s} \qquad \omega_n = 439.82 \frac{rad}{s}$$

frequency ratio:

$$r := \frac{\omega}{\omega_n} \qquad r = 0.86$$

operation near to resonance:

b) The amplitude of motion (A) for the casing and desired (Yop) for the sensing element at frequency ratio (r) are:

$$A := 0.005 \cdot in \qquad Y_{op} := 0.015 \cdot in$$

and related by:

$$\frac{Y_{op}}{A} = \left[\frac{1 + (2 \cdot \zeta \cdot r)^2}{(1 - r^2)^2 + (2 \cdot \zeta \cdot r)^2} \right]^{.5} \qquad [1]$$

Let's define G as the ratio $G := \frac{Y_{op}}{A}$ $\qquad G = 3$

From the expression for amplitude ratio: $\frac{Y_{op}}{A}$ [1]

$$G^2 \cdot \left[(1 - r^2)^2 + (2 \cdot \zeta \cdot r)^2 \right] = 1 + (2 \cdot \zeta \cdot r)^2$$

we can determine the damping ratio (ζ) from:

$$G^2 \cdot \left[(1 - r^2)^2 \right] - 1 = (2 \cdot \zeta \cdot r)^2 \cdot (1 - G^2)$$

or:

$$\zeta := \left[\frac{G^2 \cdot \left[\left(1 - r^2\right)^2\right] - 1}{(2 \cdot r)^2 \cdot \left(1 - G^2\right)}\right]^{.5}$$

Thus, at $r = 0.86$ $\zeta = 0.12$ is the damping ratio needed to produce the desired amplitude ratio. $G = 3$

c) The desired damping coefficient is equal to: $C := 2 \cdot \zeta \cdot \frac{W}{g} \cdot \omega_n$

$C = 0.14\,\text{lbf} \cdot \frac{\text{sec}}{\text{in}}$ $C = 24.91\,\text{N} \cdot \frac{\text{s}}{\text{m}}$

The stiffness of the sensor element is: $K := \omega_n^{\,2} \cdot M$

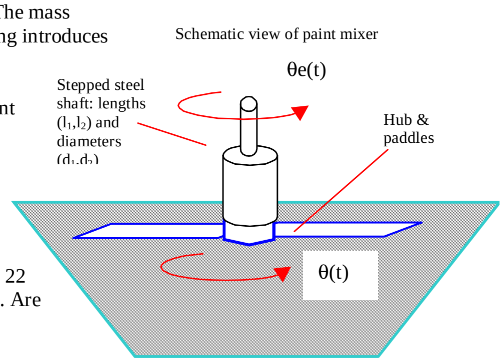
$K = 4.39 \times 10^4 \frac{\text{N}}{\text{m}}$ $K = 250.52 \frac{\text{lbf}}{\text{in}}$

MEEN 363 - Torsional vibrations

$ORIGIN := 1$

A device to mix painting is composed of the paddles and hub connected through a stepped steel shaft to an electric motor. The mass moment of inertia (***I***) of the hub and blades is 2 kg.cm^2**,** and the painting introduces a viscous damping (***D$_\theta$***) equivalent to 3 N.cm.s/rad**.** The stiffnesses of the connecting shafts equal 10 and 30 Nm/rad, respectively.

a) Derive the equation of motion for the paddles angular displacement $\theta(t)$ as a function of the motor displacement $\theta_e(t)=\Theta_m \sin(\omega t)$.
a) Calculate the system natural frequency, critical damping and damping ratio
b) Calculate & graph the FRF (amplitude and phase) of the paint mixer
c) At what motor speed the amplitude of response is the largest?
d) The engine operates at a frequency of 25 Hz with amplitude Θ_m = 22 degrees. Determine the twist angle and moment on the drive shaft. Are the calculated values reasonable (acceptable)?



a) Derive equation of motion: Sum of moments

$$\left(I \cdot \frac{d^2}{dt^2}\theta\right) = Torque_{drive} - Torque_{drag} \qquad [1]$$

$I := 2 \cdot kg \cdot cm^2$ mass moment of inertia

$$Torque_{drag} = D_\theta \cdot \frac{d}{dt}\theta$$ drag torque from viscous fluid (paint) $D_\theta := 3 \cdot N \cdot cm \cdot \frac{s}{rad}$

$$Torque_{drive} = K_\theta \cdot (\theta_e - \theta)$$ drive torque from transmission shaft

where the stiffness of the stepped shaft equals (springs in series)

$K_1 := 10 \cdot N \cdot \frac{m}{rad}$ $K_2 := 30 \cdot N \cdot \frac{m}{rad}$ $K_\theta := \left(\frac{1}{K_1} + \frac{1}{K_2}\right)^{-1}$ $K_\theta = 7.5 N \cdot \frac{m}{rad}$

Thus, the equation of motion is:

$$I \cdot \frac{d^2}{dt^2}\theta + D_\theta \cdot \frac{d}{dt}\theta + K_\theta \cdot \theta = K_\theta \cdot \theta_e = K_\theta \cdot \Theta_M \cdot \sin(\omega \cdot t) \qquad [2]$$

the natural frequency and damping ratio equal:

$\omega_n := \left(\frac{K_\theta}{I}\right)^{.5}$ $\omega_n = 193.649 \frac{rad}{s}$ $f_n := \frac{\omega_n}{2 \cdot \pi}$

$f_n = 30.82 Hz$

$\zeta := \frac{D_\theta}{2 \cdot I \cdot \omega_n}$ $\zeta = 0.387$

based on the periodic force response of a second order system, the impeller dynamic response is given by:

$$\theta(t) = \Theta_M \cdot H(r) \cdot \sin(\omega \cdot t - \phi) \qquad [3]$$

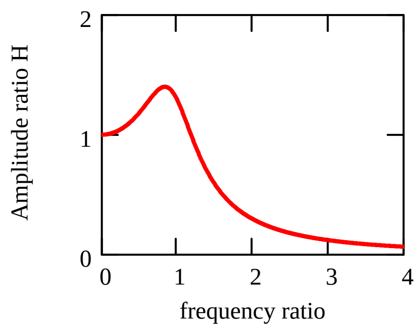
where H and ϕ are the amplification ratio and phase lag, defined as:

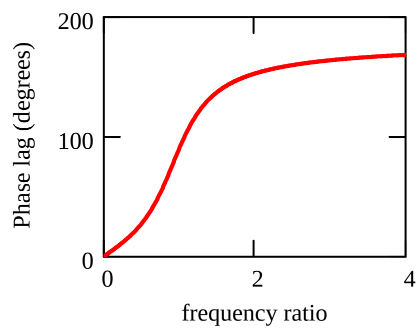
$$H(r) := \frac{1.}{\left[\left(1-r^2\right)^2 + (2\cdot\zeta\cdot r)^2\right]^{.5}}$$

$$\phi(r) := \begin{array}{|l} \phi \leftarrow \text{atan}\left(\frac{2\cdot\zeta\cdot r}{1-r^2}\right) \\ \phi \leftarrow \phi + \pi \ \text{ if } \ r > 1 \\ \text{return } \ \phi\cdot\frac{180}{\pi} \end{array}$$ [4]

with $r = \frac{\omega}{\omega_n}$ as the frequency ratio. in degrees

Graphs of the amplitude ratio and phase angle follow:





since there is a fair amount of damping, the peak amplitude of motion does **NOT** happen at r=1, i.e. when the engine frequency coincides with the natural frequency. Reviewing our knowledge, recall that the maximum amplitude occurs are (See Handout USES of FRF)

$$r_{peak} := \left(1 - 2\cdot\zeta^2\right)^{.5} \qquad r_{peak} = 0.837$$

$$H\left(r_{peak}\right) = 1.4 \qquad H_{peak} := \frac{1}{2\cdot\zeta}\cdot\frac{1}{\left(1-\zeta^2\right)^{.5}}$$

$H_{peak} = 1.4$ $\phi\left(r_{peak}\right) = 65.16$ degrees

<u>for engine operation at</u>

$f := 25\text{Hz}$ $\Theta_M := 22$ degrees

$\omega := f\cdot 2\cdot\pi$ $r := \frac{f}{f_n}$ $r = 0.811$ notice that this operating frequency is very close to the one giving the peak motion.

$H(r) = 1.398$ $\phi(r) = 61.438$ degrees

let: $\theta_r := \Theta_M\cdot H(r)$ $\theta_r = 30.753$ degrees

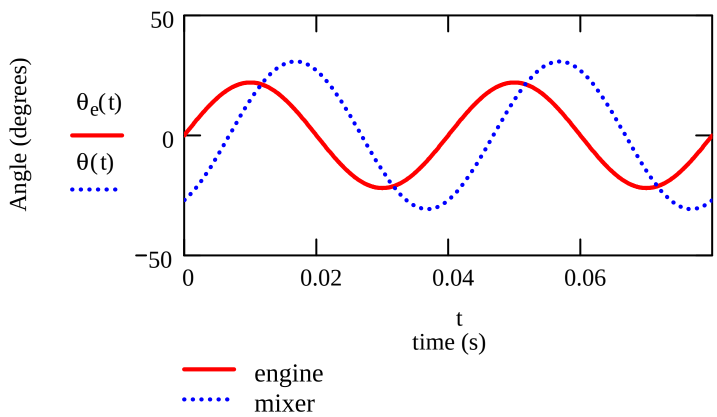
$\phi_r := \phi(r)\cdot\frac{\pi}{180}$ radians (phase angle)

Let: period of motion: $T := \frac{1}{f}$ $T = 0.04\,s$

thus, the drive forcing function and mixer responses are:

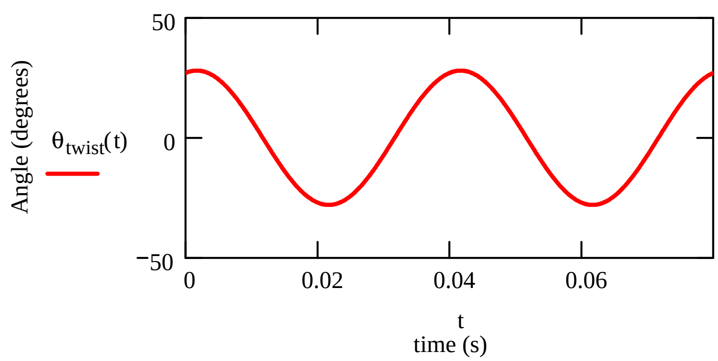
$\theta_e(t) := \Theta_M \cdot \sin(\omega \cdot t)$ $\theta(t) := \theta_r \cdot \sin(\omega \cdot t - \phi_r)$

Graph both angular responses vs. time (2 periods of motion)



$T = 0.04\,s$

the twist angle is defined as $\theta_{twist}(t) := \theta_e(t) - \theta(t)$



i.e. approximatelty equal to. $\theta_{TWIST} := 27 \cdot \frac{\pi}{180}$

The ampltude of the drive moment transmitted through the stepped shaft is just

$$Torque_{drive} := K_\theta \cdot \theta_{TWIST}$$

$$Torque_{drive} = 3.534\,N \cdot m$$

Example: Base motion excitation

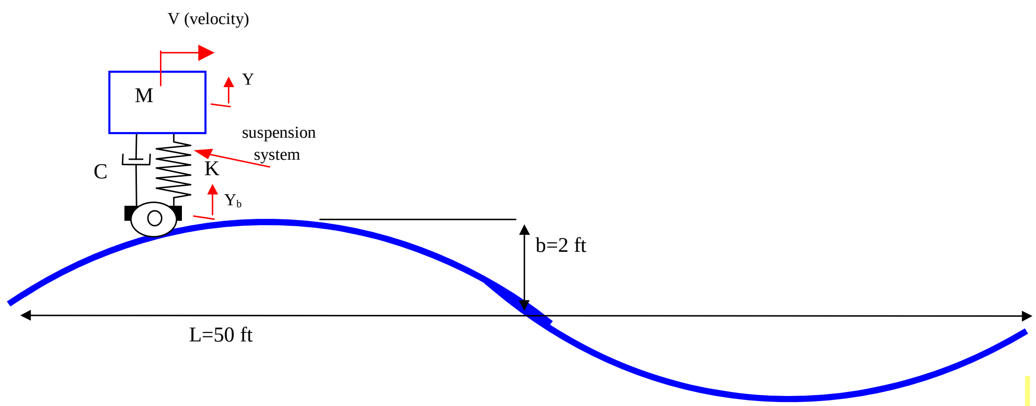
L San Andres (2007)

The figure shows a vehicle moving with speed *V* along a wavy road. Prior tests show that the vehicle weighing 2000 lbf has a natural frequency of 3 Hz. Neglect the influence of the tire's bouncing mode and determine:

a) the car speed *(V)* in mph (miles/hour) that will cause the highest amplitude of motion for the vehicle? Explain your answer

b) Find the damping ratio (ζ) and damping coefficient *(C* lin lbf.s/in*)* for the car speed in (a) such that the vehicle's absolute amplitude of motion is less than 5 ft.

c) Using *C* found in (b), calculate the vehicle steady amplitude of motion (ft) at a cruising speed *V* of 70 mph?



Data:

$M := 2000 \cdot lb$

$f_n := 3 \cdot Hz$

$\omega_n := f_n \cdot 2 \cdot \pi$

$\omega_n = 18.85 \frac{rad}{sec}$

$L := 50 \cdot ft$ path wavelength

$b := 2 \cdot ft$ amplitude of wave

KEY:

(a) the car speed must excite the car-suspension system natural frequency. The relationship between the natural period of vibration (Tn) and the time it takes the car to travel a full wave length (L) is

$mph := \frac{5280 \cdot ft}{3600 \cdot sec}$

$$\frac{L}{V} = T_n = \frac{2 \cdot \pi}{\omega_n} = \frac{1}{f_n} \qquad V_n := L \cdot f_n$$

$V_n = 150 \frac{ft}{sec}$ $\quad V_n = 102.27\, mph$

(b) The car motion amplitude (Yop) as a function of the amplitude of road wave amplitude (b), frequency ratio (r) and damping ratio ζ is (from Cheat Sheet):

$r = \frac{\omega}{\omega_n}$

$$Y_{op} = b \cdot G(r) \cos(\omega \cdot t + \phi + \gamma)$$

$$\frac{Y_{op}}{b} = G(r) = \left[\frac{1 + (2 \cdot \zeta \cdot r)^2}{\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2} \right]^{.5}$$

desired $Y_{op} := 5 \cdot ft$

Define G as $G := \frac{Y_{op}}{b}$ $\quad G = 2.5$

and working with eqn [1]:

$$G^2 \cdot \left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2 \right] = 1 + (2 \cdot \zeta \cdot r)^2$$

determine the damping ratio (ζ) from: $G^2 \cdot \left[\left(1 - r^2\right)^2\right] - 1 = (2 \cdot \zeta \cdot r)^2 \cdot \left(1 - G^2\right)$

at $r := 1$

$$\zeta := \left[\frac{G^2 \cdot \left[\left(1 - r^2\right)^2\right] - 1}{(2 \cdot r)^2 \cdot \left(1 - G^2\right)}\right]^{.5} \qquad \zeta := \left[\frac{-1}{4 \cdot \left(1 - G^2\right)}\right]^{.5} \quad [2]$$

$\zeta = 0.22$ is the damping ratio needed to produce the desired amplitude ratio, $G = 2.5$

The needed damping coefficient equals:

$$C := 2 \cdot \zeta \cdot M \cdot \omega_n \qquad C = 42.62 \, lbf \cdot \frac{sec}{in}$$

(c) At a cruising speed of

$V := 70 \cdot mph$ $\quad V = 102.67 \frac{ft}{sec}$ $\quad f_n = 3 \, Hz$

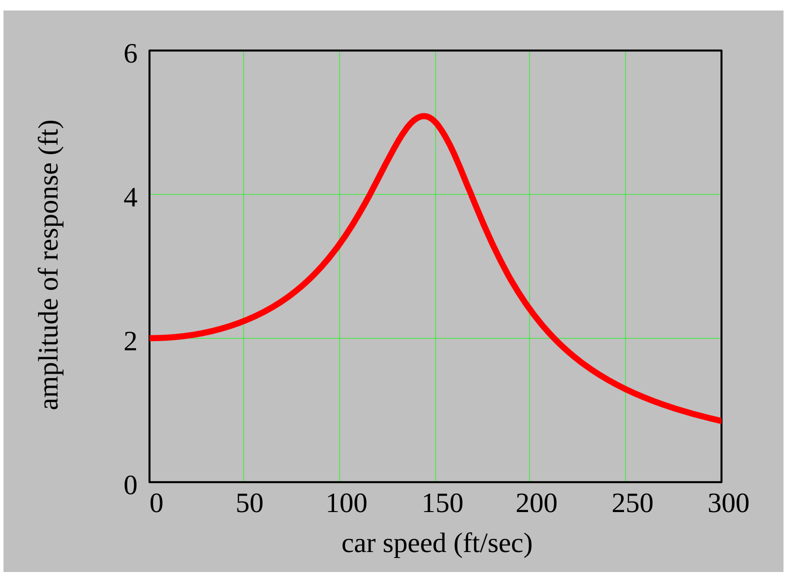
$f := \frac{V}{L}$ $\quad f = 2.05 \, Hz$ excitation frequency

$r := \frac{f}{f_n}$ $\quad r = 0.68$ frequency ratio

$$Y_{op}(r) := b \cdot \left[\frac{1 + (2 \cdot \zeta \cdot r)^2}{\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2}\right]^{.5}$$

$$Y_{op}(r) = 3.42 \, ft$$

(d) For completeness, graph the amplitude of response for multiple car speeds

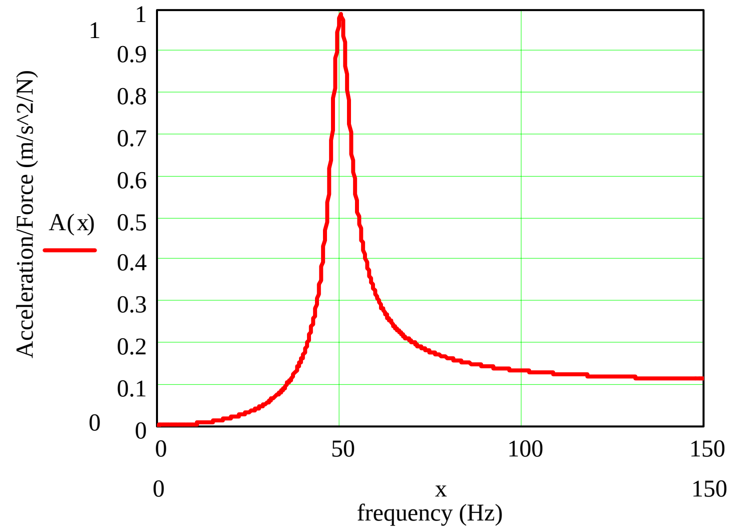


$V_n = 150 \frac{ft}{sec}$

204 mph

EXAMPLE - EXAM 2 TYPE:

Dynamic measurements were conducted on a mechanical system to determine its FRF (frequency response function). Forcing functions with multiple frequencies were exerted on the system and a digital signal analyzer (FFT) recorded the magnitude of the ACCELERATION/FORCE ([m/s2]/N) Frequency Response Function, as shown below. From the recorded data determine the system parameters, i.e. natural frequency (wn:rad/s) and damping ratio (z), and system stiffness (K:N/m), mass (M:kg), and viscous damping coefficient (C:N.s/m).
Explain procedure of ANALYSIS/INTERPRETATION of test data for full credit.



Magnitude of FRF for mechanical system

Test data showing amplitude of (acceleration/force)

Solution:

Recall that for an imposed external force of periodic form: $F(t) = F_o \cdot \sin(\omega t)$ [1]

the system response Y(t) is given by: $Y(t) = Y_{op} \cdot \sin(\omega t + \psi)$ [2]

where the amplitude of motion (Yop) and phase angle (Y) are defined as:

$$Y_{op}(r) = \frac{\frac{F_o}{K}}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \quad [3a] \qquad \Psi = -\text{atan}\left(\frac{2 \cdot \zeta \cdot r}{1 - r^2}\right) \quad [3b]$$

with $r = \frac{\omega}{\omega_n}$ [4]

from [2], we find that the acceleration is given by:

$$a_Y(t) = -\omega^2 \cdot Y_{op} \cdot \sin(\omega t + \psi) = a_{op} \cdot \sin(\omega t + \Psi - 180) \quad [5]$$

where:

$$a_{op}(r) = \frac{\frac{F_o}{M} \cdot r^2}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \quad [6]$$

since: $\frac{F_o}{K} \cdot \omega^2 = \frac{F_o}{M} \cdot \frac{\omega^2}{\omega_n^2} = \frac{F_o}{K} \cdot r^2$

thus, the magnitude of amplitude of acceleration over force amplitude follows as:

$$\frac{a_{op}(r)}{F_o} = \frac{r^2}{\left[\left(1-r^2\right)^2 + (2\cdot\zeta\cdot r)^2\right]^{.5}}\cdot\frac{1}{M} \qquad [7]$$

The units of this expression are 1/kg = $\dfrac{\frac{m}{s^2}}{N}$

For excitation at very high frequencies, r>>1.0 $\quad \dfrac{1}{M} \leftarrow \dfrac{a_{op}(r)}{F_o}$

From the graph (test data): $\quad \dfrac{1}{M} = 0.1\cdot\left(\dfrac{m}{s^2\cdot N}\right)$ $\quad$ Thus $\quad M := 10\cdot kg$

The system appears to have little damping, i.e. amplitude of FRF around a frequency of 50 Hz is rather large and varying rapidly over a narrow frequency range.

Thus, take the natural frequency as $\quad f_n := 50\cdot Hz$

expressed in rad/s as: $\quad \omega_n := f_n\cdot 2\cdot\pi \qquad \omega_n = 314.159\,\dfrac{rad}{s}$

We can estimate the stiffness (K) from the fundamental relationship:

$$K := \omega_n^{\ 2}\cdot M \qquad K = 9.87\times 10^5\,\frac{N}{m}$$

for excitation at the natural frequency (r=1), the ratio of amplitude of acceleration to force reduces to

$$\frac{a_{op}(1)}{f_o} = \frac{1}{2\cdot M\cdot\zeta}$$

from the graph (test data), the ratio is approximately equal to one (1/kg). Thus. the damping ratio is determined as

$$\zeta := \frac{1}{2\cdot M\cdot\left(\frac{1.0}{kg}\right)} \qquad \zeta = 0.05$$

That is, the system has a damping ratio equal to 5%. This result could have also been easily obtained by studying the ratio of (amplitude at the natural frequency divided by the amplitude at very high frequency, i.e.)

$$\frac{1}{2\cdot\zeta} = \frac{1}{0.1} = 10$$

Once the damping ratio is obtained, the damping coefficient can be easily determined from the formula:

$$C := \zeta\cdot 2\cdot M\cdot\omega_n \qquad C = 314.159\,N\cdot\frac{s}{m}$$

The number of calculations is minimal. One needs to interpret correctly the test data results, however.

Example: system response due to multiple frequency inputs

Consider a 2nd order system described by the following EOM

L San Andres (c) 2008

$$M\cdot\frac{d^2}{dt^2}Y + C\cdot\frac{d}{dt}Y + K\cdot Y = K\cdot z(t)$$

where

$$z(t) := a_1\cdot\cos(\omega_1\cdot t) + a_2\cdot\sin(\omega_2\cdot t) + a_3\cdot\cos(\omega_3\cdot t)$$

is an external excitation displacement function

Find the forced response of the system, i.e, find Y(t)

Given the system parameters $M := 100\cdot kg$ $K := 10^6\cdot\frac{N}{m}$ $\zeta := 0.10$

calculate natural frequency and physical damping

$$\omega_n := \left(\frac{K}{M}\right)^{0.5} \qquad f_n := \frac{\omega_n}{2\cdot\pi} \qquad f_n = 15.915\,Hz$$

$$C := 2\cdot M\cdot\omega_n\cdot\zeta \qquad C = 2\times 10^3\,s\frac{N}{m}$$

$$\omega_d := \omega_n\cdot\left(1-\zeta^2\right)^{.5} \qquad f_d := \frac{\omega_d}{2\cdot\pi} \qquad T_d := \frac{1}{f_d} \qquad T_d = 0.063\,s$$

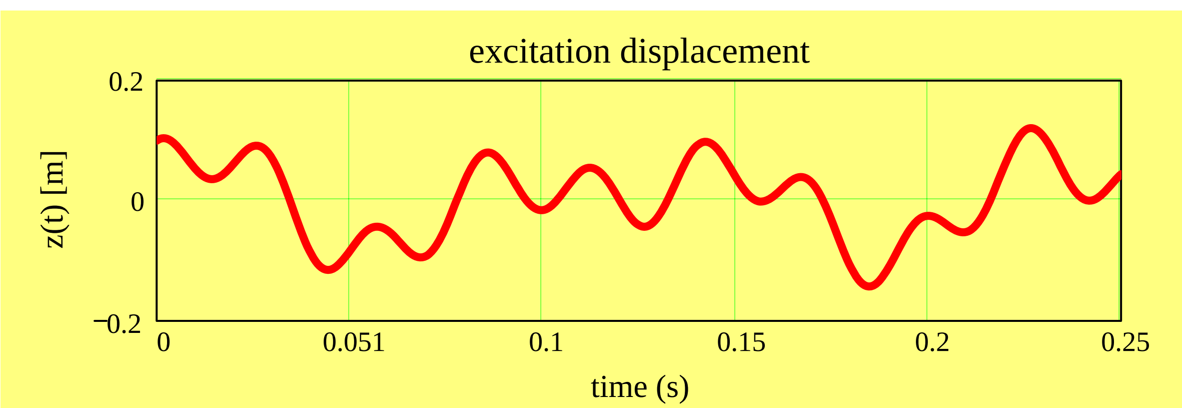
damped natural period

Set frequencies and amplitudes of the excitation z(t) are:

$$a_1 := 0.05\cdot m \qquad a_2 := 0.05\cdot m \qquad a_3 := 0.05\cdot m$$

$$\omega_1 := \frac{\omega_n}{2} \qquad \omega_2 := \omega_n\cdot\frac{9}{10} \qquad \omega_3 := 2.2\cdot\omega_n$$

assemble: $z(t) := a_1\cdot\cos(\omega_1\cdot t) + a_2\cdot\sin(\omega_2\cdot t) + a_3\cdot\cos(\omega_3\cdot t)$



4 periods of damped natural motion

SYSTEM RESPONSE is: $Y(t) := a_i \cdot H \cdot \cos(\omega_i \cdot t + \phi_i)$

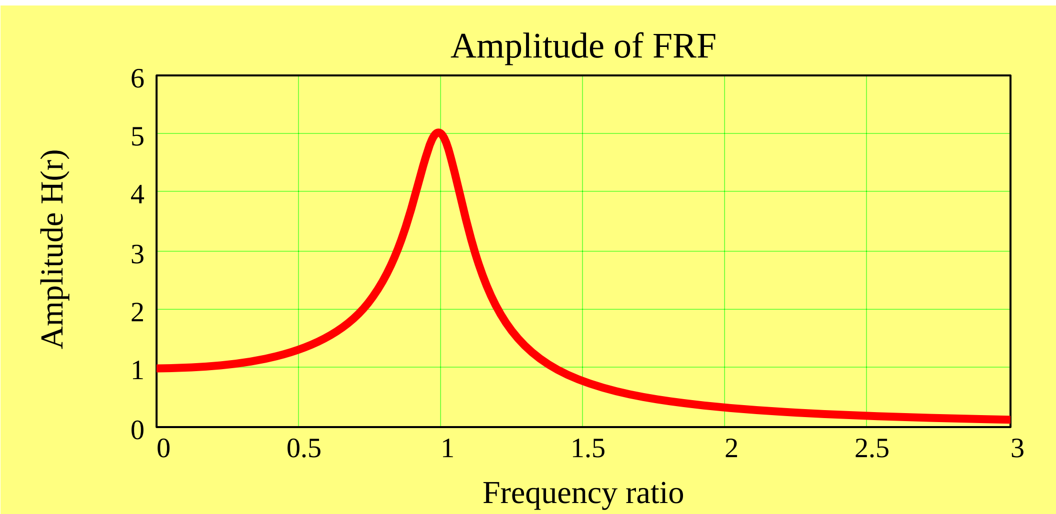
The system frequency response function: amplitude and phase angle are

$$H(r) := \frac{1}{\left[(1 - r^2)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}}$$

$$\phi(r) := \left| \begin{array}{l} \phi \leftarrow -\text{atan}\left(\frac{2 \cdot \zeta \cdot r}{1 - r^2}\right) \\ \phi \leftarrow \phi - \pi \quad \text{if} \ \ r > 1 \\ \text{return} \ \ \phi \end{array} \right.$$

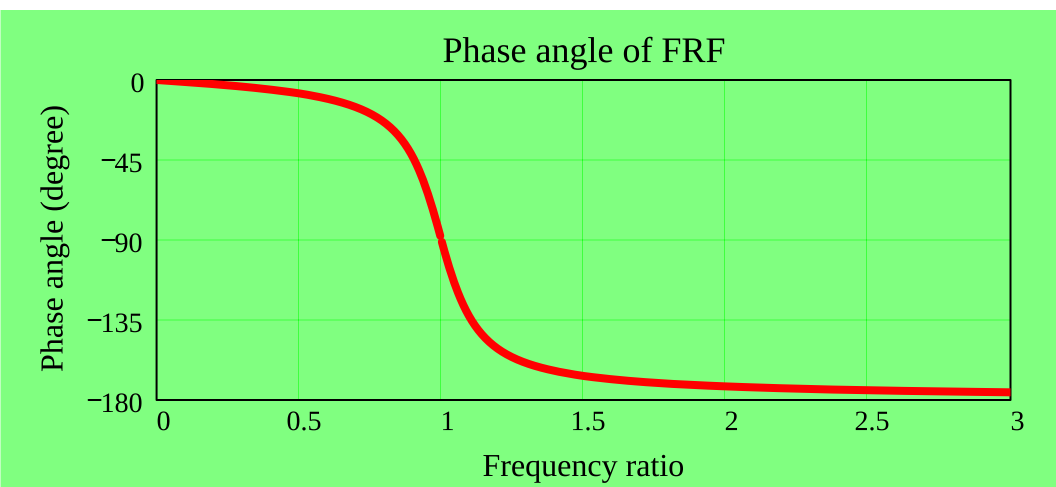
graphs of frequency response function

Amplitude and phase lag as a function of $r = \frac{\omega}{\omega_n}$ frequency ratio



$Q - \text{factor}$

$$\frac{1}{2 \cdot \zeta} = 5$$



The response of the system is given by the superposition of individual responses, i.e

$$Y(t) := Y_1 \cdot \cos(\omega_1 \cdot t + \phi_1) + Y_2 \cdot \sin(\omega_2 \cdot t + \phi_2) + Y_3 \cdot \cos(\omega_3 \cdot t + \phi_3)$$

where

for first excitation: $r_1 := \frac{\omega_1}{\omega_n}$ $H(r_1) = 1.322$ $\phi_1 := \phi(r_1)$ $r_1 = 0.5$

$Y_1 := \frac{K}{K} \cdot a_1 \cdot H(r_1)$ $Y_1 = 0.066\,m$ $\phi_1 \cdot \frac{180}{\pi} = -7.595$ degrees

for second excitation: $r_2 := \frac{\omega_2}{\omega_n}$ $H(r_2) = 3.821$ $\phi_2 := \phi(r_2)$ $r_2 = 0.9$

$Y_2 := a_2 \cdot H(r_2)$ $Y_2 = 0.191\,m$ $\phi_2 \cdot \frac{180}{\pi} = -43.452$ degrees

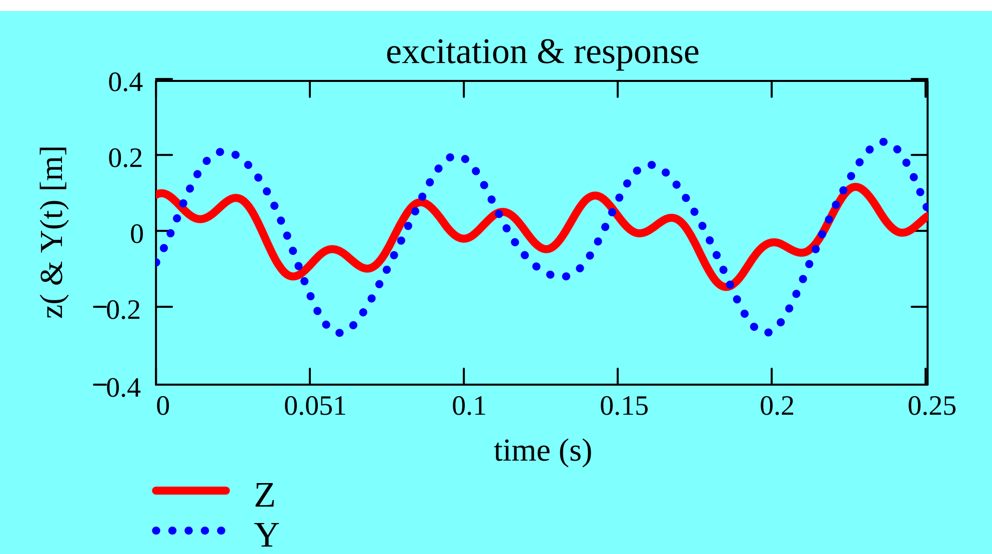
for third excitation: $r_3 := \frac{\omega_3}{\omega_n}$ $H(r_3) = 0.259$ $\phi_3 := \phi(r_3)$ $r_3 = 2.2$

$Y_3 := a_3 \cdot H(r_3)$ $Y_3 = 0.013\,m$ $\phi_3 \cdot \frac{180}{\pi} = -173.463$ degrees

Assemble physical response:

$$Y(t) := Y_1 \cdot \cos(\omega_1 \cdot t + \phi_1) + Y_2 \cdot \sin(\omega_2 \cdot t + \phi_2) + Y_3 \cdot \cos(\omega_3 \cdot t + \phi_3)$$

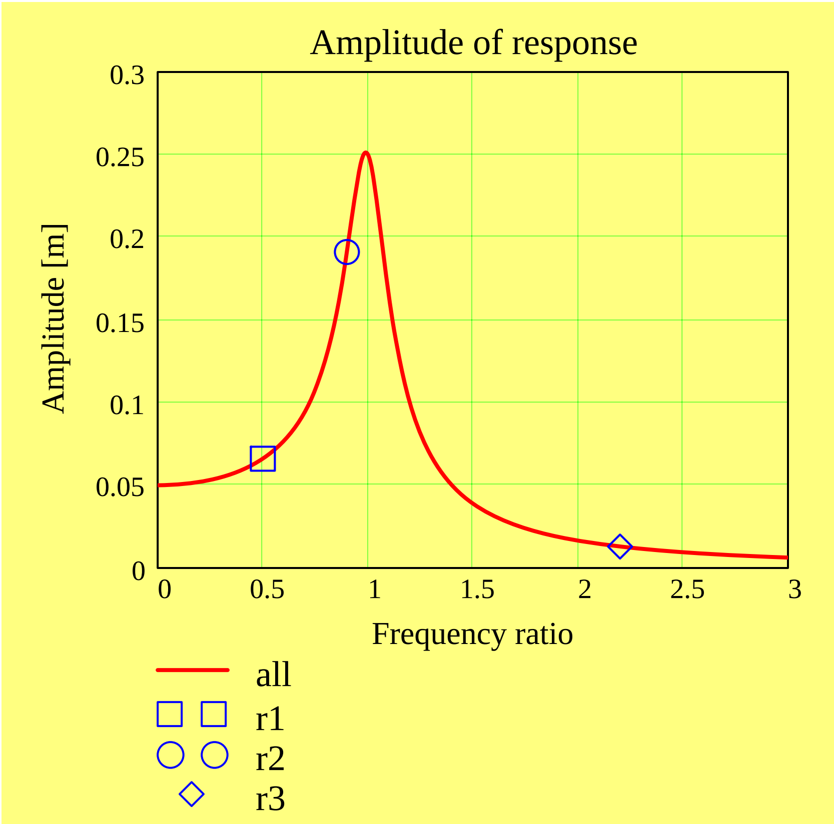
Now graph the response Y(t) and the excitation z(t):



Note: The response Y shows little motion at the highest excitation frequency (ω3). There is an obvious amplification of motion with second frequency ($\omega 2 \sim \omega n$).

To understand better, let's plot the actual FRF:

$T_d = 0.063\,s$



$a_1 = 0.05\,m$

$r_1 = 0.5$ $\quad Y_1 = 0.066\,m$

$r_2 = 0.9$ $\quad Y_2 = 0.191\,m$

$r_3 = 2.2$

$Y_3 = 0.013\,m$

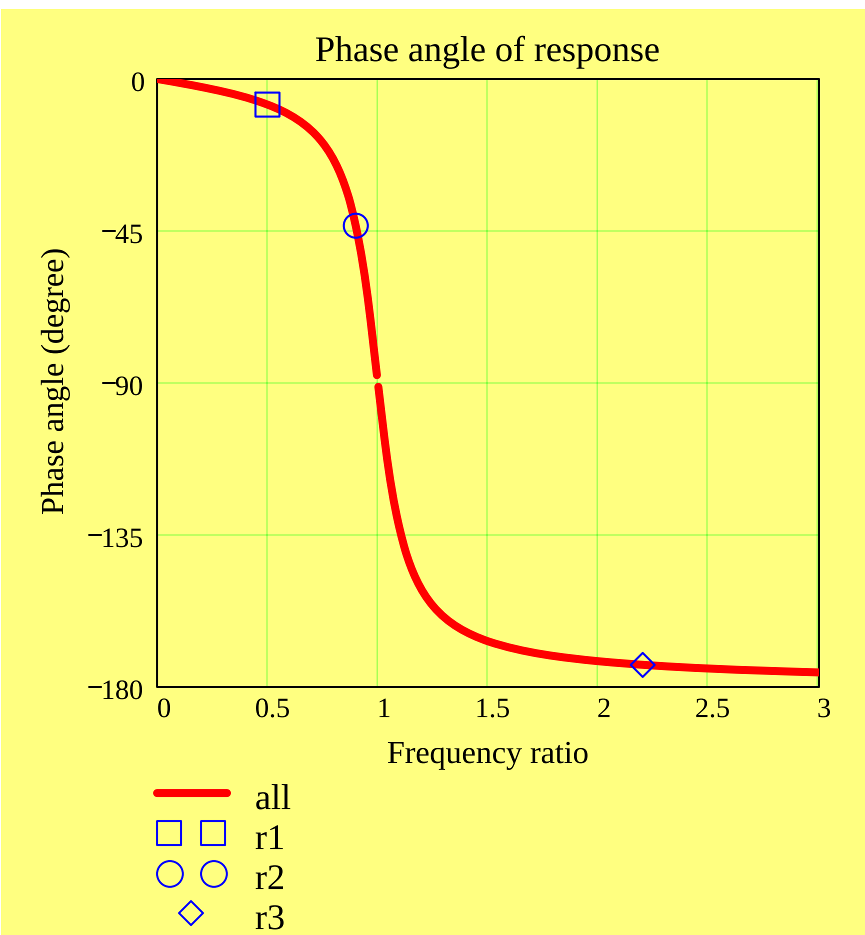
Note how response amplitude for largest frequency is largely attenuated

$Y_3 < a_3$

while amplitudes for first two frequencies are amplified, in particular for ω2 which is close to the natural frequency

$\frac{Y_2}{a_2} = 3.821$

$\frac{Y_1}{a_1} = 1.322$



Example: system response due to periodic function

Consider a 2nd order system described by the following EOM

L San Andres (c) 2008

ORIGIN := 1

$$M\cdot\frac{d^2}{dt^2}Y + C\cdot\frac{d}{dt}Y + K\cdot Y = K\cdot z(t)$$

where z (t) is an external periodic excitation function

Given the system parameters $M := 100\cdot kg$ $K := 10^6\cdot\frac{N}{m}$ $\zeta := 0.10$

calculate natural frequency and physical damping

$$\omega_n := \left(\frac{K}{M}\right)^{0.5} \qquad f_n := \frac{\omega_n}{2\cdot\pi} \qquad f_n = 15.915\,Hz$$

$$C := 2\cdot M\cdot\omega_n\cdot\zeta \qquad C = 2\times10^3\,s\frac{N}{m}$$

$$\omega_d := \omega_n\cdot\left(1-\zeta^2\right)^{.5} \qquad f_d := \frac{\omega_d}{2\cdot\pi} \qquad T_d := \frac{1}{f_d} \qquad T_d = 0.063\,s$$

damped natural period

Define periodic excitation function:

Example - square wave

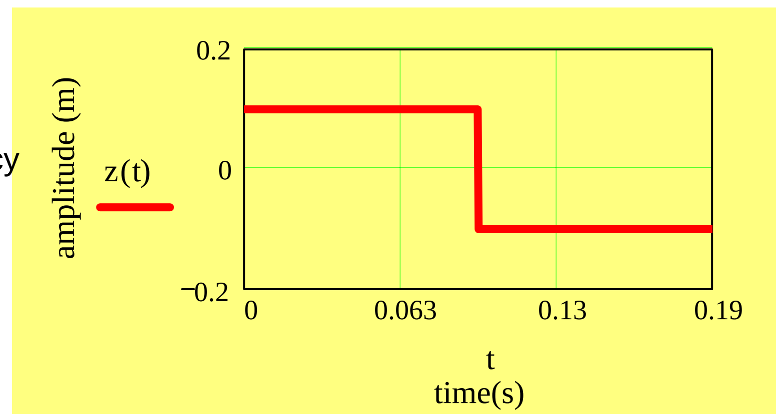
$$T := \frac{T_d}{.33333} \qquad z_o := 0.1\cdot m$$

$$z(t) := \begin{array}{|l} amp \leftarrow z_o \ \text{ if } \ t < \frac{T}{2} \\ amp \leftarrow -z_o \ \text{ if } \ t > \frac{T}{2} \\ amp \end{array}$$

$$\Omega := \frac{2\cdot\pi}{T}$$ fundamental frequency

$$\frac{\Omega}{\omega_d} = 0.333$$

$N_F := 7$ number of Fourier coefficients



Find Fourier Series coefficients for excitation z(t)

mean value $a_0 := \frac{1}{T}\cdot\int_0^T z(t)\,dt$ $\qquad a_0 = 0\,m$

$j := 1 .. N_F$

coefs of cos & sin

$$a_j := \frac{2}{T} \cdot \int_0^T z(t) \cdot \cos(j \cdot \Omega \cdot t)\, dt \qquad b_j := \frac{2}{T} \cdot \int_0^T z(t) \cdot \sin(j \cdot \Omega \cdot t)\, dt$$

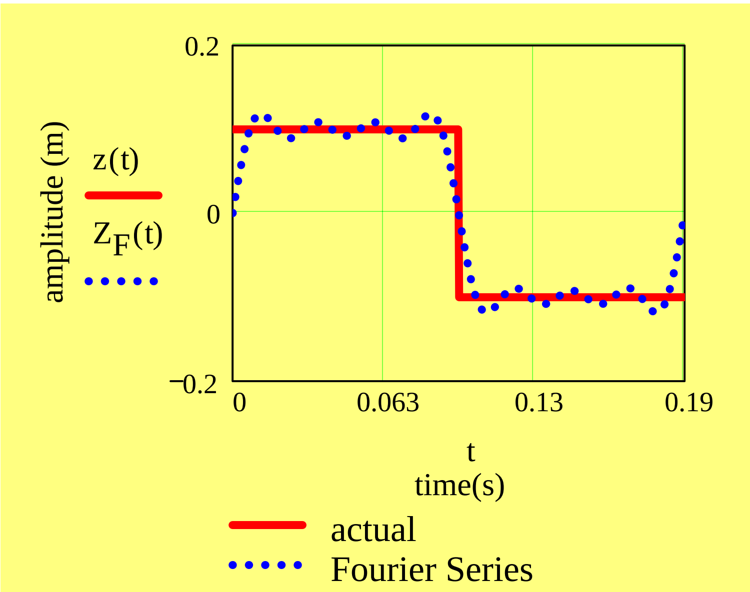
$a^T = (0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0)\,m$ $\qquad b^T = (0.127 \;\; 0 \;\; 0.042 \;\; 0 \;\; 0.025 \;\; 0 \;\; 0.018)\,m$

Build z(t) as a Fourier series

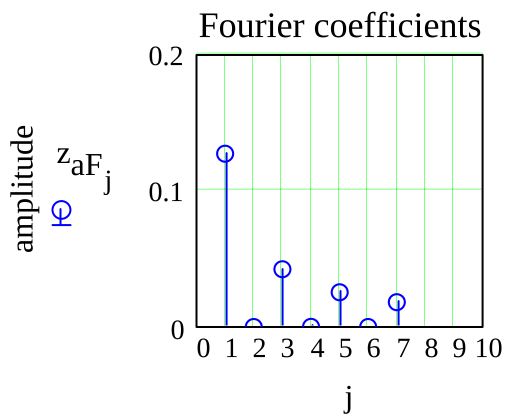
$$Z_F(t) := a_0 + \sum_{j=1}^{N_F} \left(a_j \cdot \cos(j \cdot \Omega \cdot t) + b_j \cdot \sin(j \cdot \Omega \cdot t)\right)$$

$$z_{aF_j} := \left[(a_j)^2 + (b_j)^2\right]^{.5}$$

Amplitude



$N_F = 7$



Find the forced response of the system, i.e, find Y(t)

SYSTEM RESPONSE is:

$$Y(t) := Y_o + \sum_{m=1}^{N_F} \left[\left(Yc_m \cdot \cos(m \cdot \Omega \cdot t) + Ys_m \cdot \sin(m \cdot \Omega \cdot t)\right)\right]$$

$$Y_o := a_0 \cdot \frac{K}{K}$$

$m := 1 .. N_F$

(a) set frequency ratio $\quad f_m := \frac{m \cdot \Omega}{\omega_n}$

(b) build denominator $den_m := \left[1-(f_m)^2\right]^2 + (2\cdot\zeta\cdot f_m)^2$

(c) build coefficient of cos()

$$Yc_m := \frac{K}{K}\cdot\frac{\left[a_m\cdot\left[1-(f_m)^2\right] - 2\cdot\zeta\cdot f_m\cdot b_m\right]}{den_m}$$

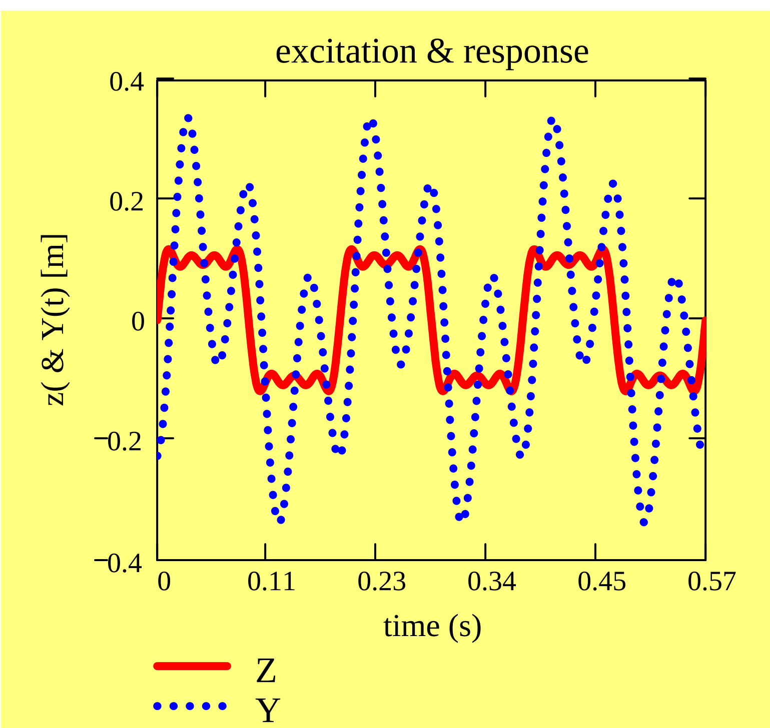
(d) build coefficient of sin()

$$Ys_m := \frac{K}{K}\cdot\frac{\left[b_m\cdot\left[1-(f_m)^2\right] + 2\cdot\zeta\cdot f_m\cdot a_m\right]}{den_m}$$

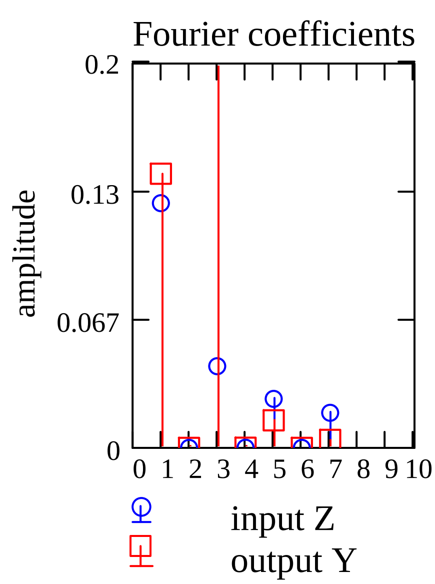
(e) for graph of components $Y_{F_m} := \left[(Yc_m)^2 + (Ys_m)^2\right]^{.5}$

$$Y(t) := Y_o + \sum_{m=1}^{N_F}\left(Yc_m\cdot\cos(m\cdot\Omega\cdot t) + Ys_m\cdot\sin(m\cdot\Omega\cdot t)\right)$$

Now graph the response Y(t) and the excitation (Fourier) z(t):



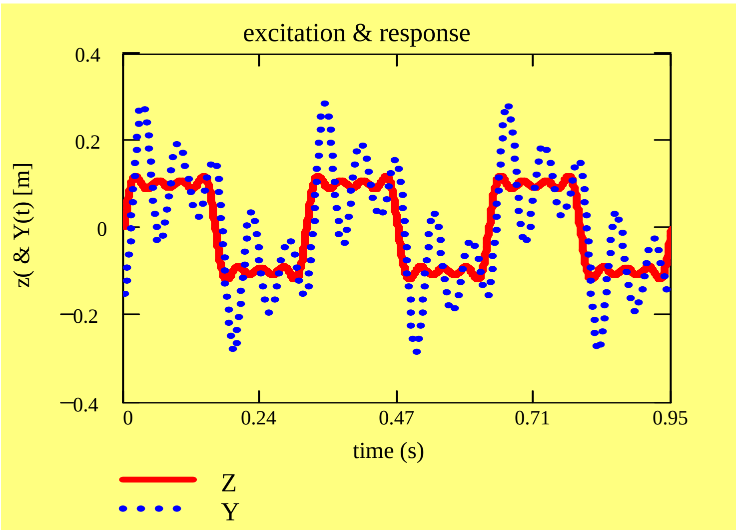
$\frac{T}{T_d} = 3$ $\frac{\Omega}{\omega_n} = 0.332$



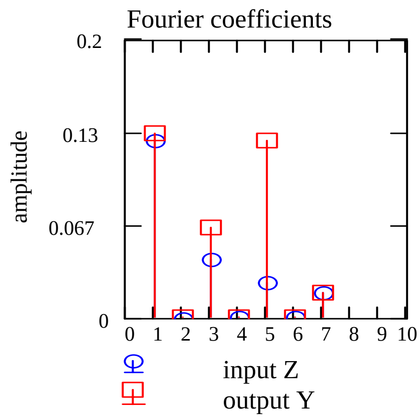
3 periods of fundamental excitation motion

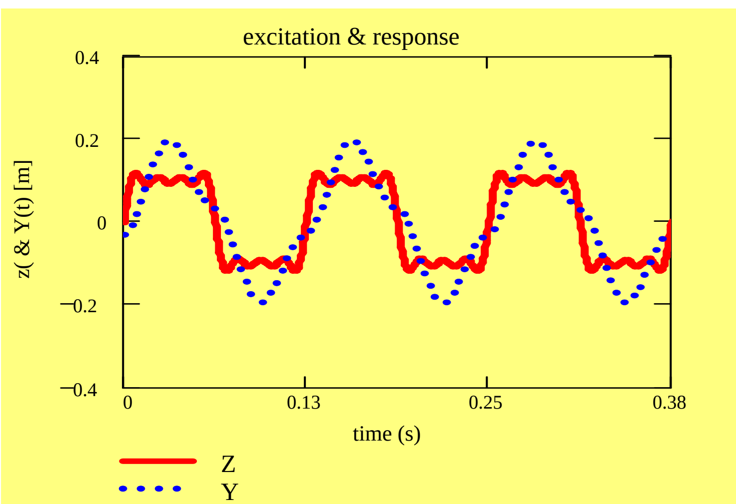
Note: obtain response for inputs with increasing frequencies (periods decrease)

==

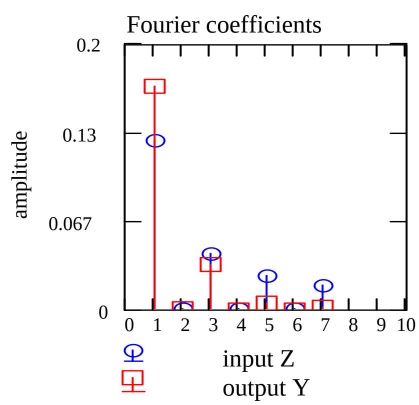


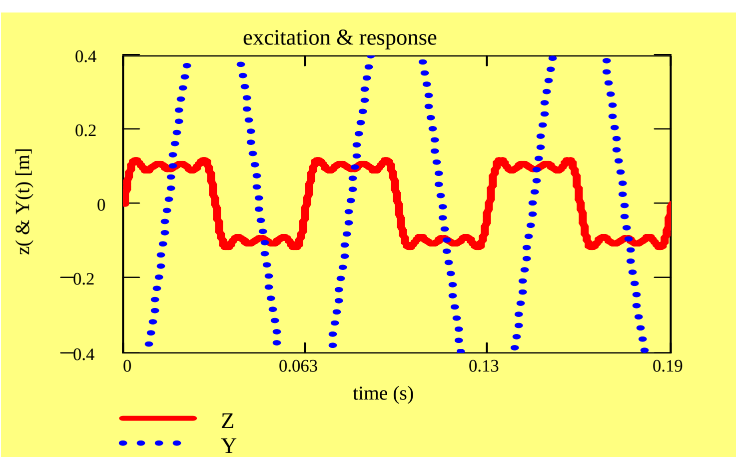
$\frac{T}{T_d} = 5$ $\frac{\Omega}{\omega_n} = 0.199$



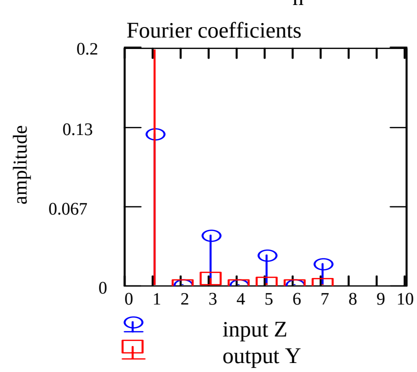


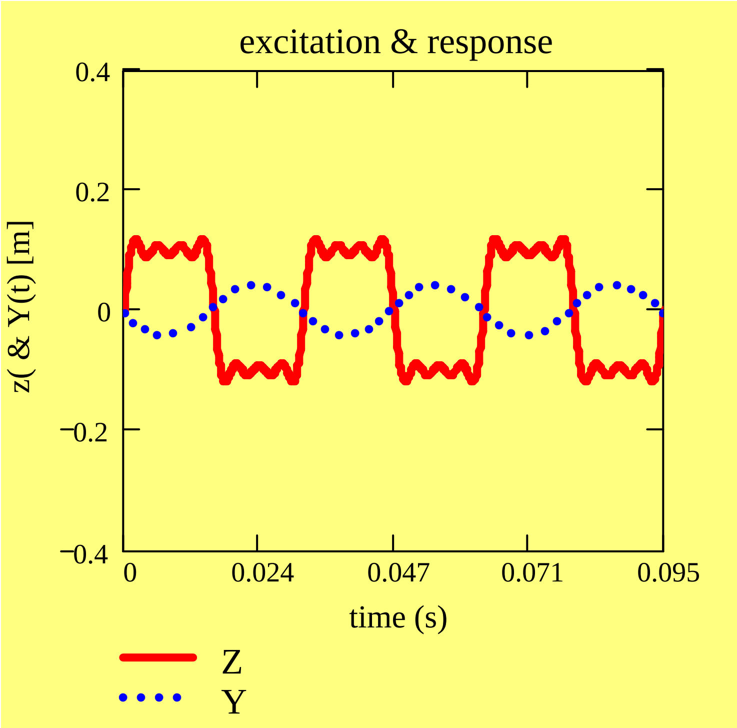
$\frac{T}{T_d} = 2$ $\frac{\Omega}{\omega_n} = 0.497$



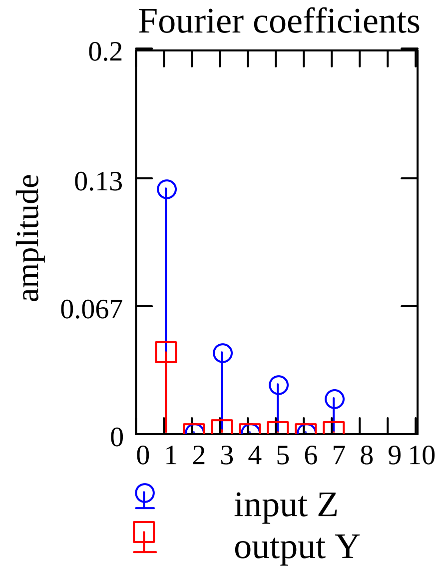


$\frac{T}{T_d} = 1$ $\frac{\Omega}{\omega_n} = 0.995$

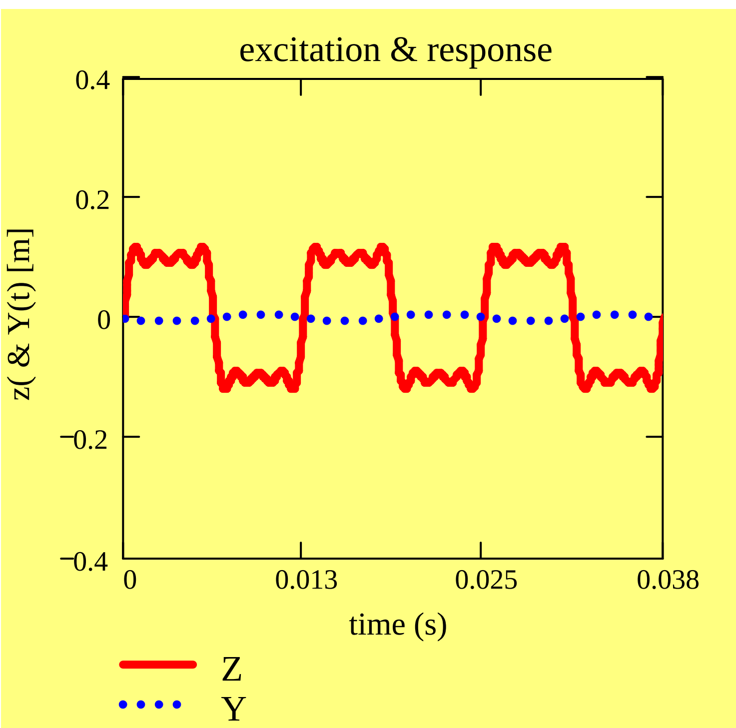




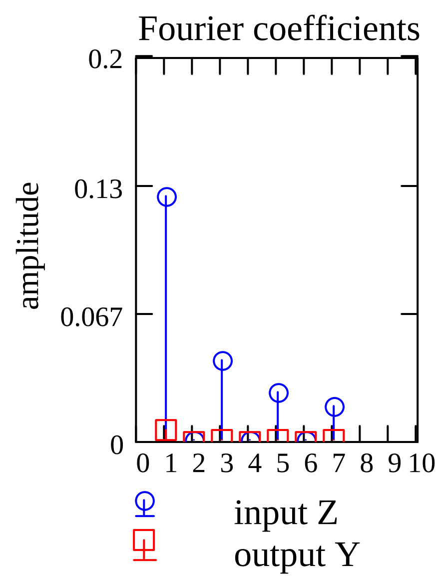
$$\frac{T}{T_d} = 0.5 \qquad \frac{\Omega}{\omega_n} = 1.99$$



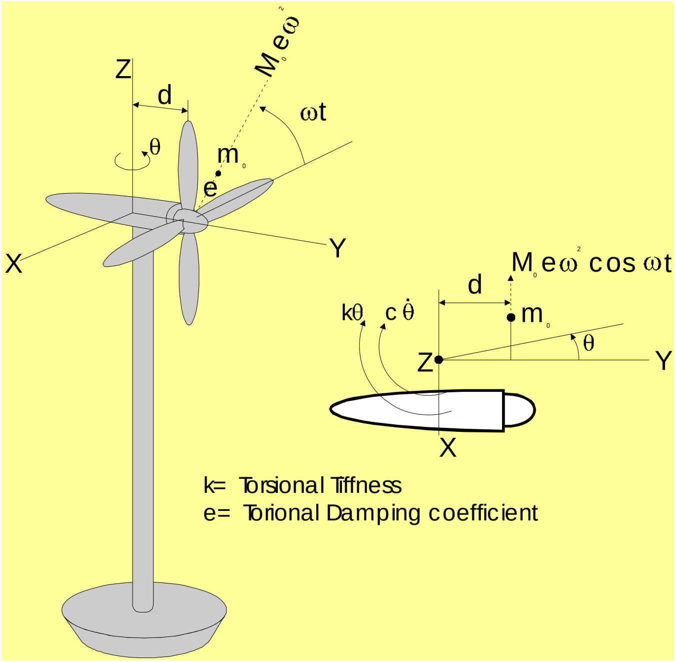
== fastest Z (smallest period)



$$\frac{T}{T_d} = 0.2 \qquad \frac{\Omega}{\omega_n} = 4.975$$



EXAMPLE TORSIONAL VIBRATIONS:



A cantilevered steel pole supports a small wind turbine. The pole torsional stiffness is ***K*** (N.m/rad) with a rotational damping coefficient ***C*** (N.m.s/rad).

The four-blade turbine rotating assembly has mass m_o, and its center of gravity is displaced distance ***e*** [m] from the axis of rotation of the assembly.

I_z (kg.m^2) is the mass moment of inertia about the *z* axis of the complete turbine, including rotor assembly, housing pod, and contents.

The total mass of the system is ***m*** (kg). The plane in which the blades rotate is located a distance ***d*** (m) from the *z* axis as shown.

For a complete analysis of the vibration characteristics of the turbine system, determine:

a) Equation of motion of the torsional vibration system about the *z* axis.
b) The steady-state torsional response $\theta_{(t)}$ (after all transients die out).
c) For system parameter values of k=98,670 N.m/rad, I_z=25 kg.m^2, C= 157 N.m.s/rad, and m_o=8 kg, e=1cm, d=30 cm, present graphs showing the response amplitude (rad) and phase angle as the turbine speed (due to wind power variations) changes from 100 rpm to 1,200 rpm.
d) From (c), at what turbine speed should the largest vibration occur and what is its magnitude?
e) Provide a design recommendation or change so as to reduce this maximum vibration amplitude value to half the original value.

Neglect any effect of the mass and bending of the pole on the torsional response, as well as any gyroscopic effects.

Note: the torque or moment induced by the mass imbalance is $T_{(\omega)} = d \times F_u = d\left(m_o\, e\ \omega^2\right)\cos(\omega t)$, i.e., a function of frequency (the rotational speed of the turbine)

The equation describing torsional motions of the turbine-pole system is:

$$I_z\, \ddot{\theta} + C\,\dot{\theta} + K\,\theta = \left(m_o e\ d\ \ \omega^2\right)\cos(\omega t) = T_{(\omega)}\ \cos(\omega t) \quad (1)$$

Note that all terms in the EOM represent moments or torques.

(b) After all transients die out, the periodic forced response of the system is

$$\theta_{(t)} = \frac{\theta_{ss}}{\left[\left(1-f^2\right)^2 + \left(2\ \zeta\ f\right)^2\right]^{1/2}}\cos\left(\omega t - \phi\right) \quad (2)$$

where $$\theta_{ss} = \frac{T_{(\omega)}}{K} = \frac{m_o\ ed\ \ \omega^2}{K} = \frac{m_o\ ed}{I_z}\ f^2 \quad (3)$$

with $$f = \frac{\omega}{\omega_n};\ \omega_n = \sqrt{\frac{K}{I_z}}\ ;\ \zeta = \frac{C}{2\sqrt{K\,I_z}},\ \text{and}\quad \phi = \tan^{-1}\left(\frac{2\zeta\ f}{1-f^2}\right) \quad (4)$$

Eq. (3) in (2) leads to

$$\theta_{(t)} = \frac{m_o\ ed}{I_z}\ \left\{\frac{f^2}{\left[\left(1-f^2\right)^2 + \left(2\ \zeta\ f\right)^2\right]^{1/2}}\right\}\cos(\omega t - \phi) \quad (5)$$

Let $\theta_\infty = \dfrac{m_o\, ed}{I_z}$ (6) & $B = \dfrac{f^2}{\left[\left(1-f^2\right)^2 + \left(2\zeta\ f\right)^2\right]^{1/2}}$ (7)

and rewrite Eq. (5) as: $$\theta(t) = \left(\theta_\infty\ B\right)\cos\ (\omega t - \phi) \quad (8)$$

(c) for the given physical values of the system parameters:

K = 98,670 N-m/rad

I_z = 25 kg m^2

C = 157 N-m/(rad/s)

$$\omega_n = \sqrt{\frac{K}{I_z}} = 62.82 \ \frac{\text{rad}}{\text{sec}}$$

$$\zeta = \frac{C}{2\sqrt{K\,I_z}} = 0.05$$

$$\theta_\infty = \frac{m_o e\, d}{I_z} = \frac{8\,kg \cdot 0.01m \cdot 0.3m}{25\,kg\, m^2} = \frac{0.024}{25} = 96\ 10^{-5}\text{rad}$$

And the turbine speed varies from 100 rpm to 1,200 rpm, i.e., ω= rpm π/30 = 10.47 rad/s to 125.66 rad/s, i.e.

$$f = \frac{\omega}{\omega_n} = \ 0.167 \ \text{ to } \ 2.00,$$

thus indicating the system will operate through resonance.

Hence, the torsion or twist angle (system response) is

$$\theta(t) = \left(96.4\ 10^{-5}\ \text{rad}\right)\cdot B\cos\ (\omega t - \phi)$$

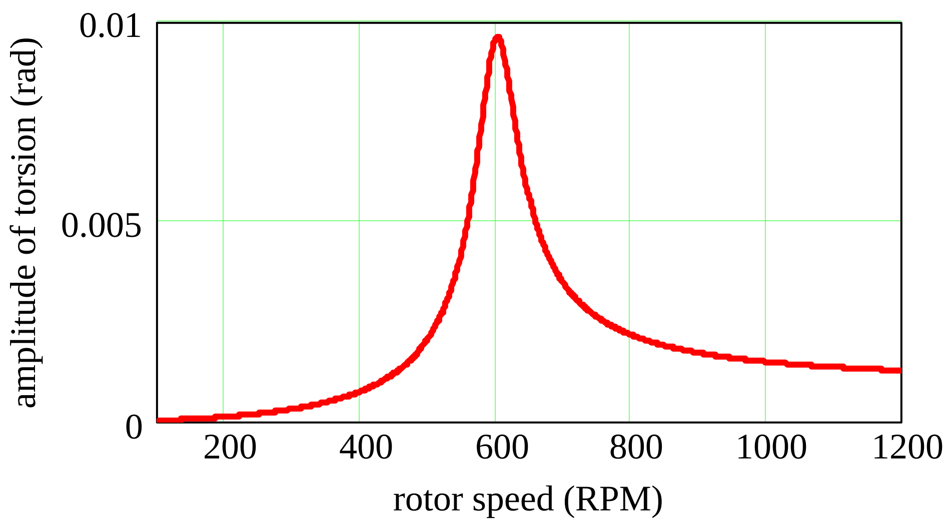
(d) Maximum amplitude of response

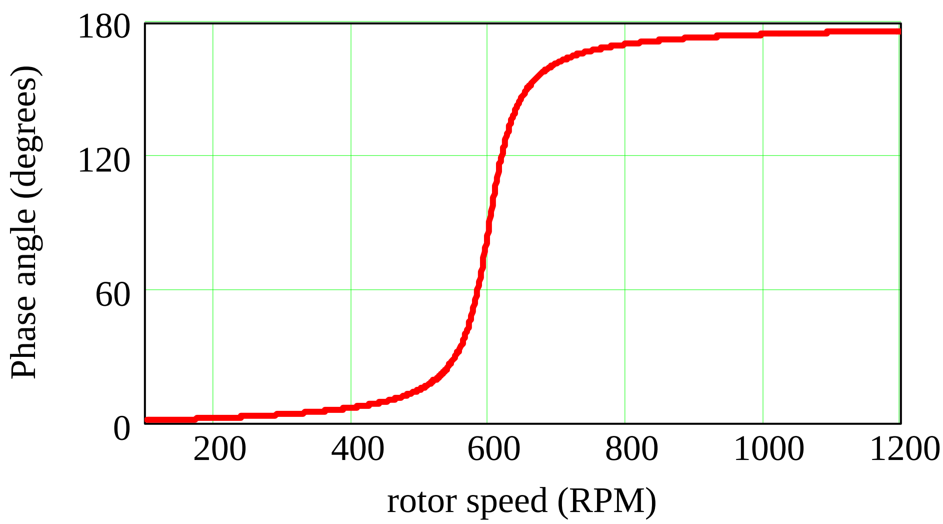
Since the damping ratio is small, $\zeta << 1$, the maximum amplitude of motion will occur when the turbine speed coincides with the natural frequency of the torsional system, i.e., at $f = \omega/\omega_n = 1$, **thus** $B \approx {}^1\!/_{2\zeta}$; and

$$\theta(t) = \theta_\infty \cdot \frac{1}{2\zeta}\cos\left(\omega t - \frac{\pi}{2}\right)$$

the magnitude is $\theta_{\max} = \frac{\theta_\infty}{2\xi} = 0.964 \text{ x } 10^{-2}$ rad**, i.e. 10 times larger than** θ_∞.

The figures below depict the **amplitude** $\left(\theta_\infty\ B\right)$ (degrees) and **phase angle** ϕ (degrees) of the pole twist as a function of the turbine rotational speed (**RPM**)





(e) Design change: DOUBLE DAMPING but first BALANCE ROTOR!