Handout #2a (pp. 1-39)

Dynamic Response of Second Order Mechanical Systems with Viscous Dissipation forces

$$M\frac{d^2X}{dt^2}+D\frac{dX}{dt}+KX=F_{ext(t)}$$

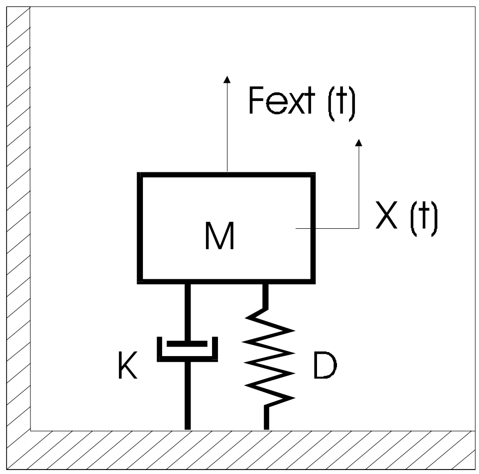
Free Response to initial conditions and *$F_{(t)}$ = 0*,

Underdamped, Critically Damped and Overdamped Systems

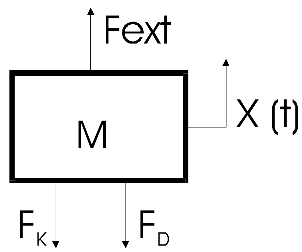
Free Response for system with Coulomb (Dry) friction

Forced Response for *Step Loading $F_{(t)} = F_o$*

Second Order Mechanical Translational System:



No dry friction (dissipation) mechanism



Free Body Diagram

<u>Fundamental equation of motion</u> (about equilibrium position, *X=0*)

$$\sum F_X = M\frac{dV}{dt} = F_{ext(t)} - F_D - F_K$$

$$F_D = DV = D\frac{dX}{dt}$$: Viscous Damping Force

$$F_k = K\,X$$: Elastic restoring Force

$$F_I = M\,a = M\frac{d^2X}{dt^2}$$: Inertia Force

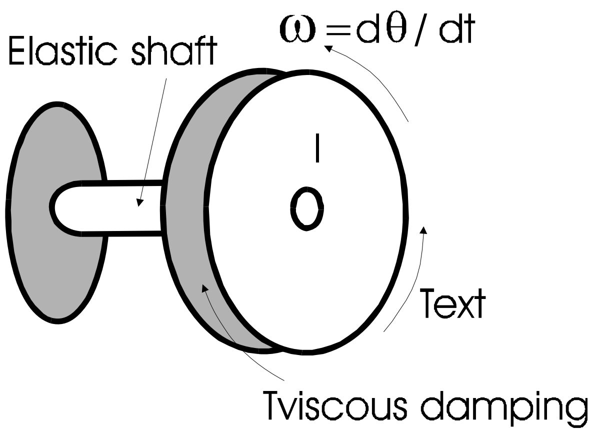
where (*M, D ,K)* represent the equivalent mass, viscous damping coefficient, and stiffness coefficient, respectively.

Since $V = \frac{dX}{dt}$ write the equation of motion as: $M\frac{d^2X}{dt^2} + D\frac{dX}{dt} + K\,X = F_{ext(t)}$

+ **Initial Conditions** in velocity and displacement; at *t=0*:

$$V(0) = V_o \text{ and } X(0) = X_o$$

Second Order Mechanical Torsional System:



Equivalent Rotational System

Fundamental equation of motion (about equilibrium position, $\theta=0$)

$$\sum Torques = I \frac{d\,\omega}{d\,t} = T_{ext(t)} - T_{\theta D} - T_{\theta K}$$

$T_{\theta D} = D_\theta \omega$: Viscous dissipation torque

$T_{\theta K} = K_\theta \theta$: Elastic restoring torque

$T_{\theta I} = I\, d\omega / dt$: Inertia torque

where (I, D_θ, K_θ) are equivalent mass moment of inertia, rotational viscous damping coefficient, and rotational (torsional) stiffness coefficient, respectively.

Since $\omega = d\theta/dt$, then write equation of motion as:

$$I \frac{d^2\theta}{d\,t^2} + D_\theta \frac{d\,\theta}{d\,t} + K_\theta\, \theta = T_{ext(t)}$$

+ Initial Conditions in angular velocity and displacement at $t=0$:

$$\omega(0) = \omega_o \quad \text{and} \quad \theta(0) = \theta_o$$

(a) **Free Response of Second Order Mechanical System**

Pure Viscous Damping Forces

Let the external force be null *($F_{ext}=0$)* and consider the system to have an initial displacement X_o and initial velocity V_o. The equation of motion for a 2nd order system with viscous dissipation is:

$$M\frac{d^2X}{dt^2}+D\frac{dX}{dt}+K\,X=0 \tag{1}$$

with initial conditions $V(0)=V_o$ and $X(0)=X_o$

Divide Eq. (1) by *M* and define:

$\omega_n=\sqrt{K/M}$: undamped natural frequency of system

$\zeta=\dfrac{D}{D_{cr}}$: viscous damping ratio,

where $D_{cr}=2\sqrt{K\,M}$ is known as the critical damping value

With these definitions, Eqn. (1) becomes:

$$\frac{d^2X}{dt^2}+2\zeta\,\omega_n\frac{dX}{dt}+\omega_n^2\,X=0 \tag{2}$$

The solution of the Homogeneous Second Order Ordinary Differential Equation with Constant Coefficients is of the form:

$$X(t)=A\,e^{st} \tag{3}$$

Where *A* is a constant yet to be found from the initial conditions.

Substitute Eq. (3) into Eq. (2) and obtain:

$$\left(s^2 + 2\zeta\,\omega_n\,s + \omega_n^2\right)A = 0 \tag{4}$$

Note that *A* must be different from zero for a non trivial solution. Thus, Eq. (4) leads to the **CHARACTERISTIC EQUATION** of the system given as:

$$\left(s^2 + 2\zeta\,\omega_n\,s + \omega_n^2\right) = 0 \tag{5}$$

The roots of this 2nd order polynomial are:

$$s_{1,2} = -\zeta\,\omega_n \mp \omega_n\left(\zeta^2 - 1\right)^{1/2} \tag{6}$$

The nature of the roots (eigenvalues) clearly depends on the value of the damping ratio ζ . Since there are two roots, the solution to the differential equation of motion is now rewritten as:

$$X(t) = A_1\,e^{s_1 t} + A_2\,e^{s_2 t} \tag{7}$$

where A_1, A_2 are constants determined from the initial conditions in displacement and velocity.

From Eq. (6), differentiate three cases:

Underdamped System: $0 < \zeta < 1, \rightarrow D < D_{cr}$

Critically Damped System: $\zeta = 1, \rightarrow D = D_{cr}$

Overdamped System: $\zeta > 1, \rightarrow D > D_{cr}$

Note that $\tau = \left(1/\zeta\omega_n\right)$ has units of time; and for practical purposes, it is regarded as an equivalent time constant for the second order system.

Free Response of Undamped 2nd Order System

For an undamped system, ζ = *0*, i.e. a conservative system without viscous dissipation, the roots of the characteristic equation are imaginary:

$$s_1 = -i\,\omega_n \;;\; s_2 = i\,\omega_n \qquad (8)$$

where $i = \sqrt{-1}$ is the imaginary unit.

Using the complex identity $e^{iat} = \cos(at) + i\,\sin(at)$, renders the **undamped response** of the conservative system as:

$$X(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t) \qquad (9.a)$$

where $\omega_n = \sqrt{K/M}$ is the **natural frequency** of the system.

At time *t* = *0*, the initial conditions are $V(0) = V_o$ and $X(0) = X_o$

hence
$$C_1 = X_0 \text{ and } C_2 = \frac{V_0}{\omega_n} \qquad (9.b)$$

and equation (9.a) can be written as:

$$X(t) = X_M \cos(\omega_n t - \varphi) \qquad (9.c)$$

Where $X_M = \sqrt{X_0^2 + \frac{V_0^2}{\omega_n^2}}$ and $\tan(\varphi) = \frac{V_0}{\omega_n\, X_0}$

X_M is the **maximum amplitude response.**

Notes:

In a purely conservative system, the motion never dies out, it is harmonic and periodic.
Motion always oscillates about the equilibrium position ***X* = *0***

Free Response of Underdamped 2nd Order System

For an underdamped system, $0 < \zeta < 1$, the roots are complex conjugate (real and imaginary parts), i.e.

$$s_{1,2} = -\zeta\,\omega_n \mp i\,\omega_n \left(1-\zeta^2\right)^{1/2} \qquad (10)$$

where $i = \sqrt{-1}$ is the imaginary unit.

Using the complex identity $e^{iat} = \cos(at) + i\,\sin(at)$, write the solution for **underdamped response** of the system as:

$$X(t) = e^{-\zeta\,\omega_n\,t}\left(C_1 \cos\left(\omega_d t\right) + C_2 \sin\left(\omega_d t\right)\right) \qquad (11)$$

where $\omega_d = \omega_n \left(1-\zeta^2\right)^{1/2}$ is the system damped natural frequency.

At time $t = 0$, the initial conditions are $V(0) = V_o$ and $X(0) = X_o$

Then $$C_1 = X_0 \;\text{ and }\; C_2 = \frac{V_0 + \zeta\,\omega_n X_0}{\omega_d} \qquad (11.b)$$

Equation (11) representing the system response can also be written as:

$$X(t) = e^{-\zeta\,\omega_n\,t} X_M \cos\left(\omega_d t - \varphi\right) \qquad (11.c)$$

where $X_M = \sqrt{C_1^2 + C_2^2}$ and $\tan\left(\varphi\right) = \dfrac{C_2}{C_1}$

Note that as $t \to \infty$, $X_{(t)} \to 0$, i.e. the equilibrium position only if $\zeta > 0$;

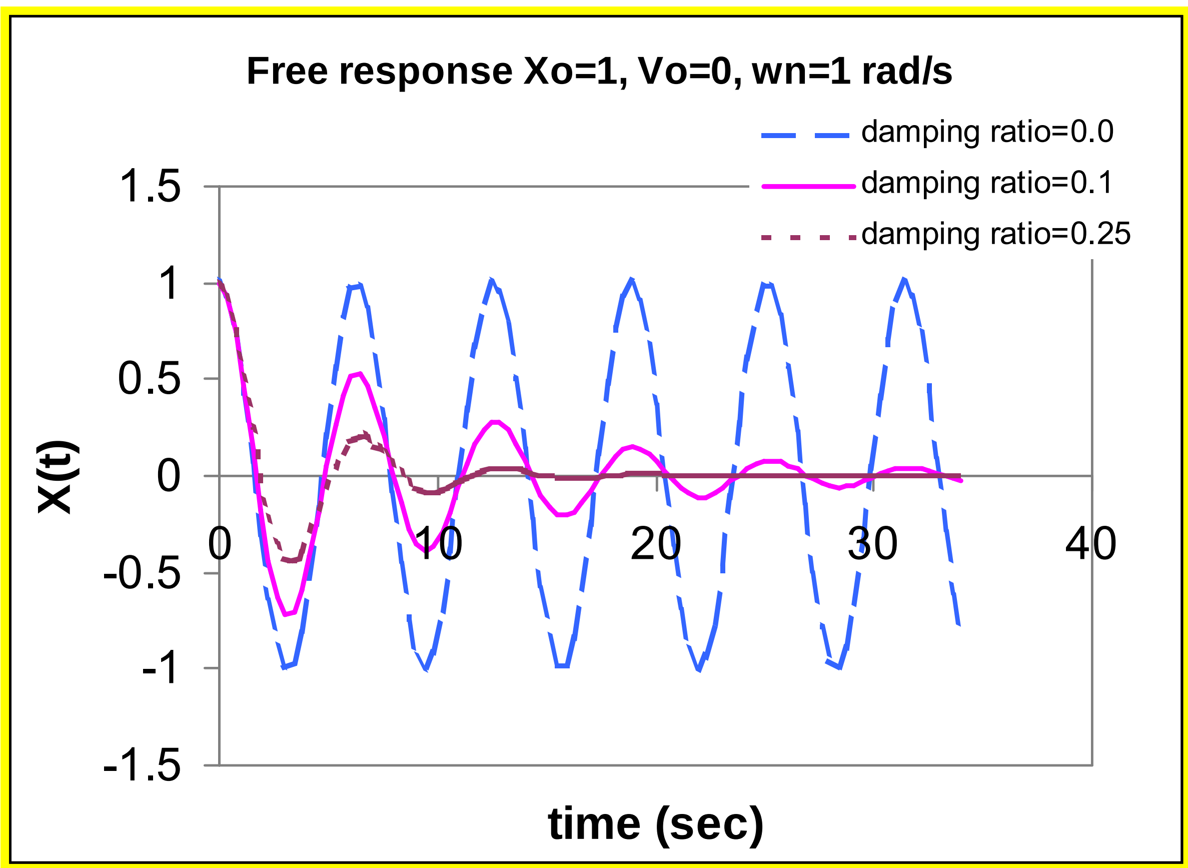
and X_M is the **largest amplitude of response** only if $\zeta = 0$ *(no damping).*

Free Response of Underdamped 2nd Order System:
initial displacement only damping ratio varies

$X_o = 1, \; V_o = 0, \quad \omega_n = 1.0 \; rad/s \quad \zeta = 0, \; 0.1, \; 0.25$

Motion decays exponentially for $\zeta > 0$

Faster system response as ζ increases, i.e. faster decay towards equilibrium position $X=0$



Free Response of Underdamper 2nd Order System:

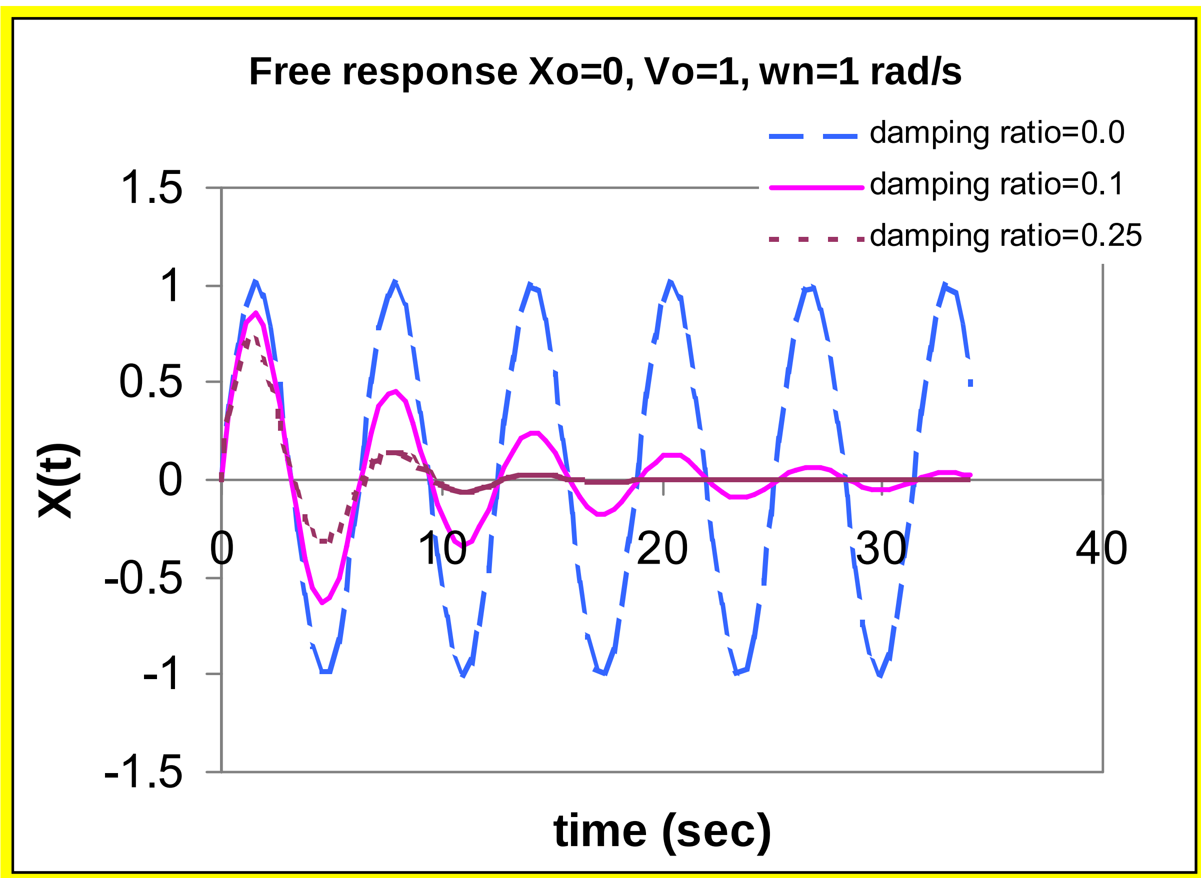
Initial velocity only damping varies

$X_o = 0, \; V_o = 1.0 \qquad \omega_n = 1.0 \; rad/s; \qquad \zeta = 0, \; 0.1, \; 0.25$

Motion decays exponentially for $\zeta > 0$

Faster system response as ζ increases, i.e. faster decay towards equilibrium position $X=0$

Note the initial overshoot



Free Response of Overdamped 2nd Order System

For an overdamped system, ζ > *1*, the roots of the characteristic equation are real and negative, i.e.

$$s_1 = \omega_n \left[-\zeta + \left(\zeta^2 - 1 \right)^{1/2} \right]; \; s_2 = \omega_n \left[-\zeta - \left(\zeta^2 - 1 \right)^{1/2} \right] \qquad (12)$$

The **overdamped free response** of the system as:

$$X(t) = e^{-\zeta \omega_n t} \left(C_1 \cosh\left(\omega_* t\right) + C_2 \sinh\left(\omega_* t\right) \right) \qquad (13)$$

where $\omega_* = \omega_n \left(\zeta^2 - 1 \right)^{1/2}$ has units of 1/time. Do not confuse this term with a frequency since motion is NOT oscillatory.

At time *t* = *0*, the initial conditions are $V(0) = V_o$ and $X(0) = X_o$

Then

$$C_1 = X_0 \;\text{ and }\; C_2 = \frac{V_0 + \zeta \omega_n X_0}{\omega_*} \qquad (14)$$

Note that as $t \to \infty$, $X_{(t)} \to 0$, i.e. the equilibrium position.

An **overdamped system does to oscillate**. The larger the damping ratio ζ >*1*, the longer time it takes for the system to return to its equilibrium position.

Free Response of Critically Damped 2nd Order System

For a critically damped system, ζ = *1*, the roots are real negative and identical, i.e.

$$s_1 = s_2 = -\zeta\,\omega_n \qquad (15)$$

The solution form *X(t) = A e**st*** is no longer valid. For repeated roots, the theory of ODE's dictates that the family of solutions satisfying the differential equation is

$$X(t) = e^{-\omega_n t}\left(C_1 + t\,C_2\right) \qquad (16)$$

At time *t* = *0*, the initial conditions are $V(0) = V_o$ and $X(0) = X_o$

Then $$C_1 = X_0 \;\text{ and }\; C_2 = V_0 + \omega_n X_0 \qquad (17)$$

Note that as $t \to \infty$, $X_{(t)} \to 0$, i.e. the equilibrium position.

A **critically damped system** does to oscillate, and it is the fastest to damp the response due to initial conditions.

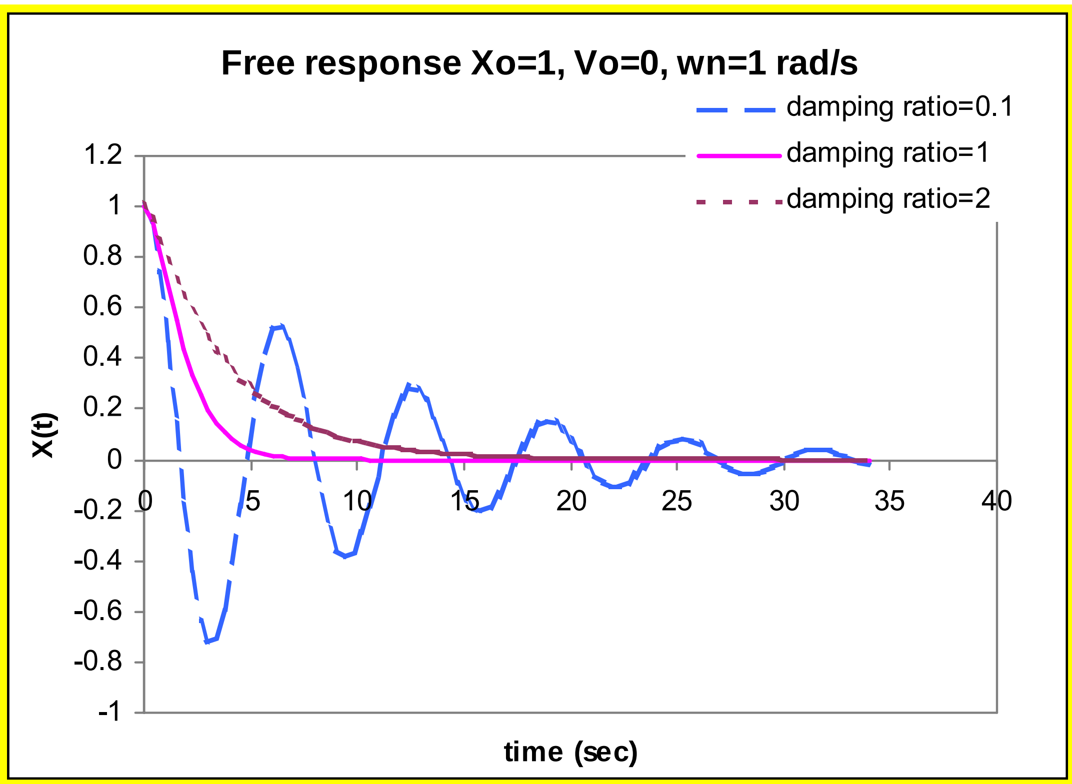
Free Response of 2nd order system:

Comparison between underdamped, critically damped and overdamped systems

initial displacement only

$X_o = 1, \; V_o = 0 \quad \omega_n = 1.0 \; rad/s \quad \zeta = 0.1, \; 1.0, \; 2.0$

Motion decays exponentially for $\zeta > 0$
Fastest response for $\zeta = 1$; i.e. fastest decay towards equilibrium position $X = 0$



Free Response of 2nd order System:

Comparison between underdamped, critically damped and overdamped systems

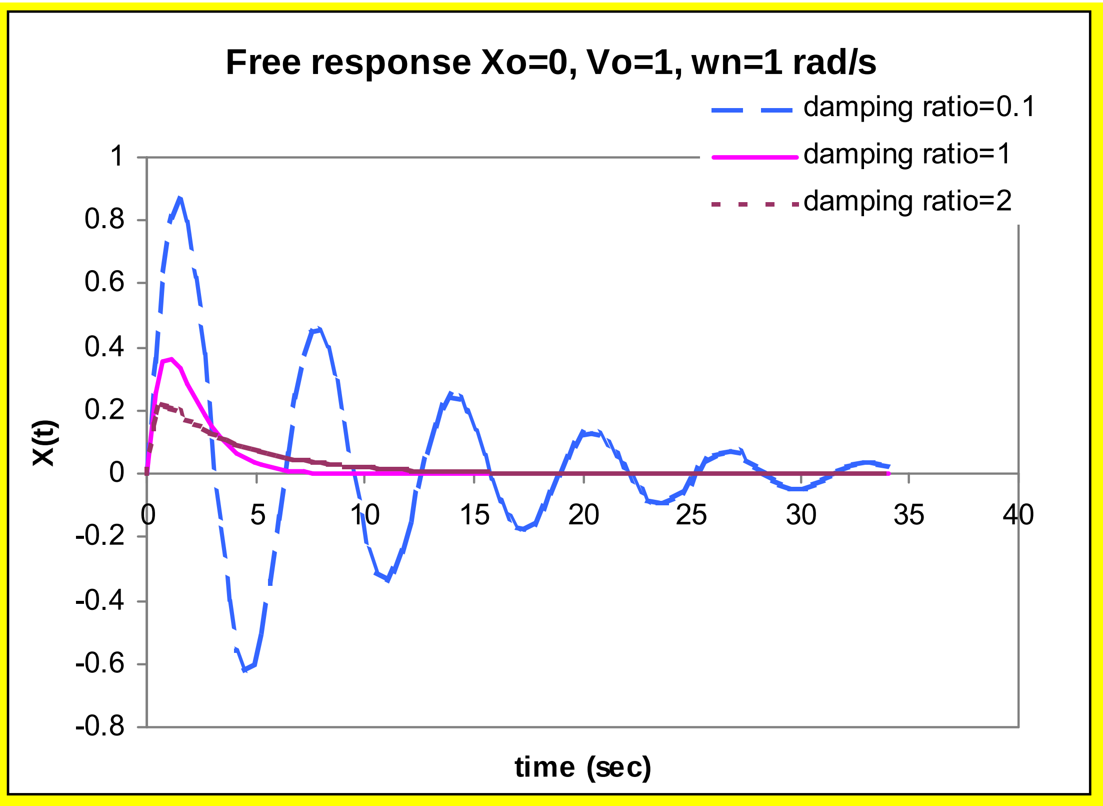
Initial velocity only

$X_o = 0$, $V_o = 1.0$, $\omega_n = 1.0$ rad/s ζ = 0.1, 1.0, 2.0

Motion decays exponentially for $\zeta > 0$

Fastest response for $\zeta = 1.0$, i.e. fastest decay towards equilibrium position $X=0$.

note initial overshoot



Free Response of 2nd order System:

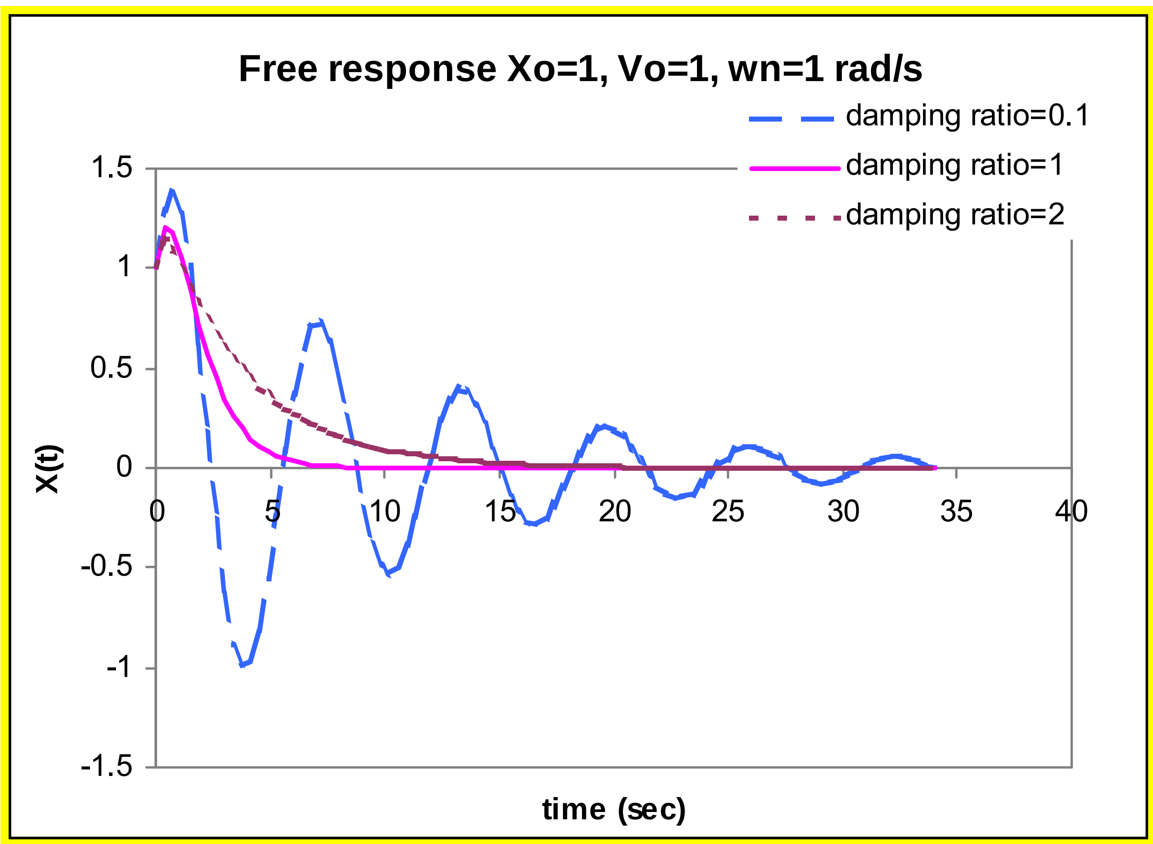
Comparison between underdamped, critically damped and overdamped systems

initial displacement and velocity

$X_o = 1, \ V_o = 1 \quad \omega_n = 1.0 \ rad/s \quad \zeta = 0.1, \ 1.0, \ 2.0$

Motion decays exponentially for $\zeta > 0$

Fastest decay to equilibrium position $X = 0$ for $\zeta = 1.0$



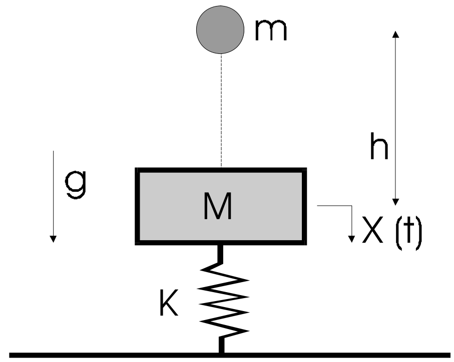
E X A M P L E:

A 45 gram steel ball (m) is dropped from rest through a vertical height of h=2 m. The ball impacts on a solid steel cylinder with mass M = 0.45 kg. The impact is perfectly elastic. The cylinder is supported by a soft spring with a stiffness K = 1600 N/m. The mass-spring system, initially at rest, deflects a maximum equal to δ = 12 mm, from its static equilibrium position, as a result of the impact.

(a) Determine the time response motion of the mass-spring system.

(b) Sketch the time response of the mass-spring system.

(c) Calculate the height to which the ball will rebound.



(a) Conservation of linear momentum before impact = just after impact:

$$mV_{-} = mV_{+} + M\,\dot{x}_o \quad (1)$$

where $V_{-} = \sqrt{2gh}$ = 6.26 m/s is the steel ball velocity before impact

V_{+} = velocity of ball after impact; and $\dot{x}_o$: initial mass-spring velocity.

Mass-spring system EOM: $M\,\ddot{x} + K\,x = 0$ (2) with $\omega_n = \sqrt{\frac{K}{M}}$ = 59.62 rad/s

(from static equilibrium), the initial conditions are $x(0) = 0$ and $\dot{x}(0) = \dot{x}_o$ (3)

(2) & (3) lead to the undamped free response: $x(t) = \frac{\dot{x}_o}{\omega_n}\sin(\omega_n t) = \delta\sin(\omega_n t)$

given δ = 0.012 m as the largest deflection of the spring-mass system.

Hence, $\dot{x}_o = \delta\,\omega_n$ = 0.715 m/s

(c) Ball velocity after impact: from Eq. (1))

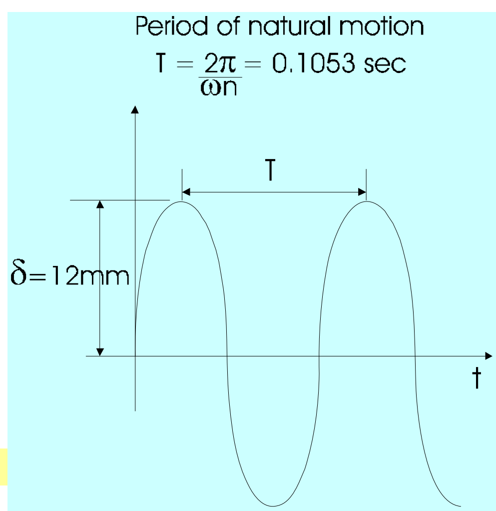
$$V_{+} = V_{-} - \frac{M}{m}\dot{x}_o = (6.26 - 7.15)\frac{m}{s} = -0.892\ \frac{m}{s}$$

(upwards)

and the height of rebound is

$$h_{+} = \left[\frac{V_{+}^2}{2g}\right] = 41 \text{ mm}$$

(b) Graph of motion

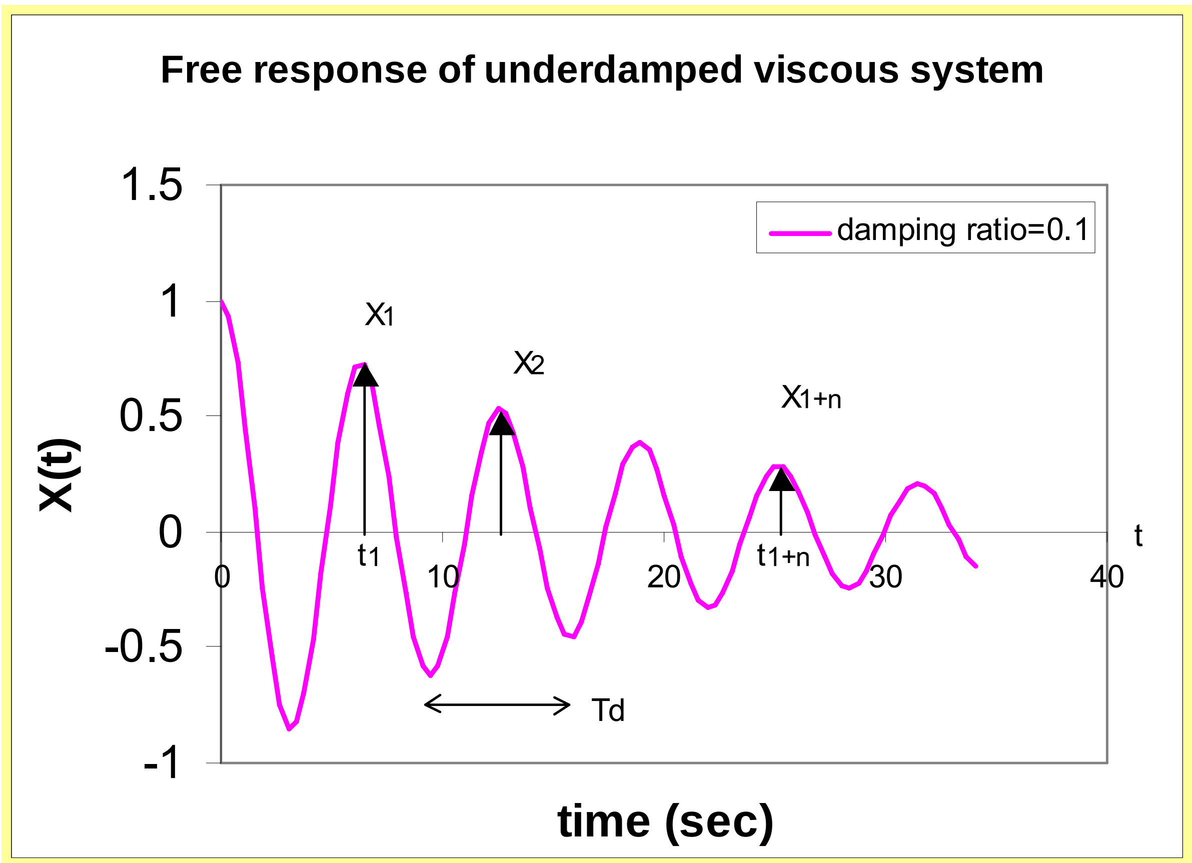


The concept of **logarithmic decrement** for estimation of the viscous damping ratio from a free-response vibration test

The free vibration response of an **underdamped** 2nd order viscous system (M,K,D) due to an initial displacement X_o is a decay oscillating wave with damped natural frequency (ω_d). The period of motion is $T_d = 2\pi/\omega_d$ (sec). The free vibration response is

$$x(t) = X_o \, e^{-\zeta\omega_n t} \cos\left(\omega_d t\right) \qquad (1)$$

where $\zeta = D/D_{cr}$, $D_{cr} = 2\sqrt{KM}$; $\omega_n = (K/M)^{1/2}$; $\omega_d = \omega_n\left(1-\xi^2\right)^{1/2}$



Consider two peak amplitudes, say X_1 and X_{1+n}, separated by $\boldsymbol{n}$ periods of decaying motion. These peaks occur at times, t_1 and

(t_1+nTd), respectively. The system response at these two times is from Eq. (1):

$$X_1 = x(t_1) = X_o e^{-\zeta\,\omega_n t_1} \cos\,(\omega_d t_1),$$

and

$$X_{1+n} = x\left(t_{1+nT_d}\right) = X_o\, e^{-\zeta\omega_n (t_1+nT_d)} \cos\,(\omega_d\, t_1 + n\,\omega_d T_d)\,,$$

Or, since $\omega_d T_d = 2\pi$.

$$X_{1+n} = X_o e^{-\zeta\,\omega_n (t_1+nT_d)} \cos\left(\omega_d t_1 + 2\pi\, n\right) = X_o e^{-\zeta\,\omega_n (t_1+nT_d)} \cos\,(\omega_d t_1), \tag{2}$$

Now the ratio between these two peak amplitudes is:

$$\frac{X_1}{X_{1+n}} = \frac{\left\{X_o\, e^{-\zeta\omega_n t_1} \cos\,(\omega_d t_1)\right\}}{\left\{X_o\, e^{-\zeta\omega_n (t_1+nT_d)} \cos\,(\omega_d t_1 + 2\pi)\right\}} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1+nT_d)}} = e^{\zeta\omega_n (nT_d)} \tag{3}$$

Take the natural logarithm of the ratio above:

$$\ln\left(X_1 / X_{1+n}\right) = \xi\,\omega_n\, n\, T_d = n\,\zeta\,\omega_n \frac{2\pi}{\omega_n \left(1-\zeta^2\right)^{1/2}} = \frac{2\pi\, n\,\zeta}{\left(1-\zeta^2\right)^{1/2}} = n\cdot\delta \tag{4}$$

and, define the **logarithmic decrement** as:

$$\delta = \frac{1}{n}\ln\left[\frac{X_1}{X_{1+n}}\right] = \frac{2\pi\,\zeta}{\left(1-\zeta^2\right)^{1/2}} \tag{5}$$

Thus, the ratio between peak response amplitudes determines a useful relationship to identify the damping ratio of an underdamped second order system, i.e. once the **log dec** (δ) is determined then,

$$\zeta = \frac{\delta}{\left[(2\pi)^2 + \delta^2 \right]^{1/2}} \tag{6}$$

and for small damping ratios, $\zeta \sim \dfrac{\delta}{2\pi}$.

The **logarithmic decrement** method to identify viscous damping ratios should only be used if:

a) the time decay response shows an oscillatory behavior (i.e. vibration) with a clear exponential envelope, i.e. damping of viscous type,
b) the system is linear, 2nd order and **underdamped**,
c) the dynamic response is very clean, i.e. without any spurious signals such as noise or with multiple frequency components,
d) the dynamic response $X(t) \rightarrow 0$ as $t \rightarrow \infty$. Sometimes measurements are taken with some DC offset. This must be removed from your signal before processing the data.
e) Use more than just two peak amplitudes separated *n* periods. In practice, it is more accurate to plot the magnitude of several peaks in a semi-log paper and obtain the log-decrement *(δ)* as the best linear fit to the following relationship [see below Eq. (7)].

From equation (2),

$$X_{1+n} = X_o \cos\left(\omega_d t_1\right) e^{-\zeta\,\omega_n\left(t_1 + nT_d\right)} = X_1 e^{-\xi\omega_n\left(nT_d\right)},$$

where $X_1 = X_o \cos\left(\omega_d t_1\right) e^{-\zeta\,\omega_n\left(t_1\right)}$

$$\ln(X_{1+n}) = \ln X_1 + \ln(e^{-\zeta\,\omega_n(nT_d)}) =$$

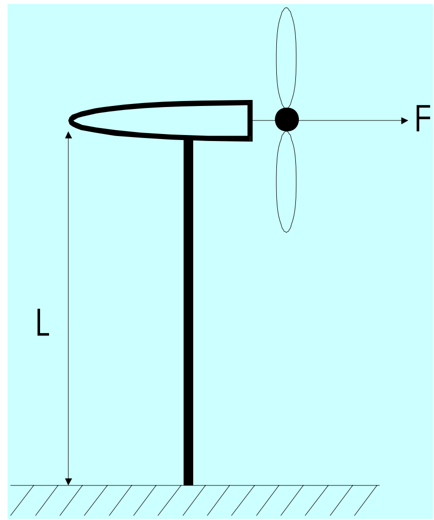
$$\ln X_1 - \zeta\,\omega_n T_d = A - n\,\delta, \text{ where } A = \ln X_1;$$

$$\ln(X_{1+n}) = A - n\,\delta \qquad (7)$$

i.e., plot the natural log of the peak magnitudes versus the period numbers (*n=1,2,…)* and obtain the logarithmic decrement from a straight line curve fit. In this way you will have used more than just two peaks for your identification of damping.

Always provide the correlation number (goodness of fit = r^2) for the linear regression curve $(y=ax+b)$, where $y=ln(X)$ and $x=n$ as variables

EXAMPLE



A wind turbine is modeled as a concentrated mass (the turbine) atop a weightless elastic tower of height L. To determine the dynamic properties of the system, a large crane is brought alongside the tower and a lateral force F=200 lb is exerted along the turbine axis as shown. This causes a horizontal displacement of 1.0 in.

The cable attaching the turbine to the crane is instantaneously severed, and the resulting free vibration of the turbine is recorded. At the end of two complete cycles (periods) of motion, the time is 1.25 sec and the motion amplitude is 0.64 in.

From the data above determine:

(a) equivalent stiffness K (lb/in)

(b) damping ratio ζ

(c) undamped natural frequency ω_n (rad/s)

(d) equivalent mass of system (lb-s^2/in)

a) $$K = \frac{\text{static force}}{\text{static deflection}} = \frac{200\ lb}{1.0\text{ in}} = 200\ \text{lb/in}$$

b) Use **log dec** to find the viscous damping ratio

cycle	amplitude	time
0	1.0 in	0.0 sec
2	0.64 in	1.25 sec

$$\delta = \frac{1}{n}\ \ln\left(\frac{x_o}{x_2}\right) = \frac{1}{2}\ln\left(\frac{1.0}{0.64}\right) = 0.2231$$

$$\delta = \frac{2\pi\zeta}{\sqrt{1-{}^2}}\ ; \qquad \zeta = \frac{\delta}{\sqrt{\delta^2 + 4\pi^2}} \simeq \frac{\delta}{2\pi} = 0.035$$

underdamped system with 3.5% of critical damping.

c) Damped period of motion, $T_d = \frac{1.25}{2} \frac{\text{s}}{\text{cyc}} = 0.625$ **sec/cyc**

Damped natural frequency, $$\omega_d = \frac{2\pi}{T_d} = 10.053 \frac{rad}{\text{sec}}$$

Natural frequency, $$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \mathbf{10.059} \frac{\textbf{rad}}{\textbf{sec}}$$

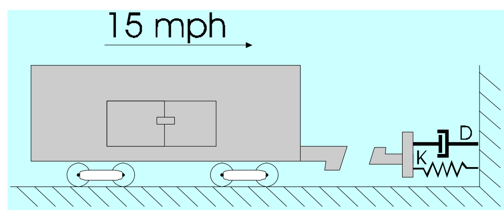
d) Equivalent mass of system:

from $\omega_n = \sqrt{K/M}$

$$M = \frac{K}{\omega_n^{\ 2}} = \frac{200\ lb/\text{in}}{10.059^2 \quad 1/\text{sec}^2}$$ = **1.976 lb/ sec²/in**

EXAMPLE

A loaded railroad car weighing 35,000 lb is rolling at a constant speed of 15 mph when it couples with a spring and dashpot bumper system. If the recorded displacement-time curve of the loaded railroad car after coupling is as shown, determine



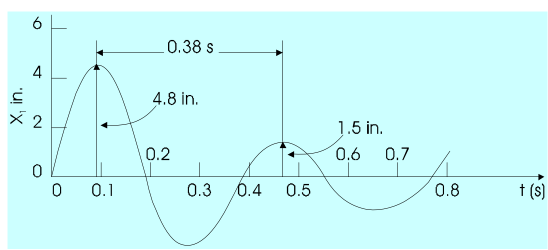
(a) the logarithmic decrement δ

(b) the damping ratio ζ

(c) the natural frequency ω_n (rad/sec)

(d) the spring constant K of the bumper system (lb/in)

(e) the damping ratio ζ of the system when the railroad car is empty. The unloaded railroad car weighs 8,000 lbs.



(a) logarithmic decrement $$\delta = \ell n\left(\frac{x_o}{x_1}\right) = \ell n\left(\frac{4.8}{1.5}\right) = 1.1631$$

(b) damping ratio

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \Rightarrow \zeta = \frac{\delta}{\left[(2\pi)^2 + \delta^2\right]^{1/2}}$$

$\zeta \equiv 0.1820$

(c) Damped natural period: T_d **= 0.38 sec.**

and frequency

$$\omega_d = \frac{2\pi}{T_d} = 16.53 \frac{rad}{\sec} = \omega_n \sqrt{1-\zeta^2}$$

The natural frequency is

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 16.816 \frac{rad}{\sec}$$

(d) Bumper stiffness,

$$K = \omega_n^2 \ \mathrm{M_{car}} = 16.816^2 \frac{1}{\sec^2} \left[\frac{35{,}000 \ \ \mathrm{lb}}{386.4 \ \mathrm{in/sec^2}}\right] = K = 25612.5 \frac{lb}{in}$$

(e) Damping ratio when car is full: $\zeta = \dfrac{D}{2\sqrt{K \ \mathrm{M_{full}}}} = 0.182$

Note that the physical damping coefficient (D) does not change whether car is loaded or not, but ζ does change.

Damping ratio when car is empty $\zeta_e = \dfrac{D}{2\sqrt{K \ M_{empty}}}$

The ratio

$$\frac{\zeta}{\zeta_e} = \sqrt{\frac{M_{empty}}{M_{full}}} \Rightarrow \zeta_e = \zeta \sqrt{\frac{M_{full}}{M_{empty}}} = 0.182 \left[\frac{35{,}000}{8{,}000}\right]^{1/2}$$

$\zeta_e \equiv 0.381$

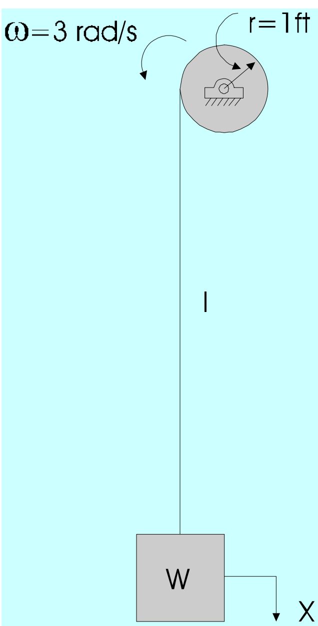
EXAMPLE:

An elevator weighing 8000 lb is attached to a steel cable that is wrapped around a drum rotating with a constant angular velocity of 3 rad/s. The radius of the drum is 1 ft. The cable has a net cross-sectional area of 1 in^2 and an effective modulus of elasticity E = 12 $(10)^6$ psi. A malfunction in the motor drive system of the drum causes the drum to stop suddenly when the elevator is moving down and the length ℓ of the cable is 50 ft. Neglect damping and determine the maximum stress in the cable.
(Coordinate *X(t)* describes motion after cable stops, recall that the elastic cable is already stretched due to the elevator weight before the cable stops).

The cable stiffness is just $K = \frac{AE}{L} = 20,000 \frac{lb}{in}$ and $M = \frac{8000\ lb}{386\ in/sec^2} = 20.70 \frac{lb \cdot sec^2}{in}$, and

$$\omega_n = 31.08 \ \frac{\text{rad}}{\text{sec}^2}$$

At the time of stop, the eqn. of motion is $$M\ddot{X} + K\,X = 0 \qquad (1)$$



+ I.C. *X(o)=0,* and $\dot{X}(0) = \omega\, r = 3 \text{ ft/sec} = 36 \text{ in/sec}$

The motion [soln. of (1)] is: $$X(t) = \frac{\dot{X}_o}{\omega_n} \sin(\omega_n t). \qquad (2)$$

With

maximum Dynamic Displacement is: $$X_d = \frac{\dot{X}_o}{\omega_n} \qquad (3)$$

One can also obtain (3) from conservation of mechanical energy

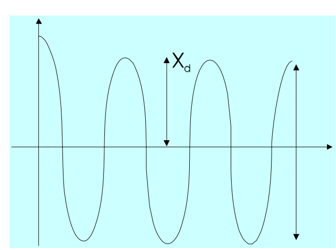
$T_{max} = V_{max}$

$$\frac{1}{2} M \dot{X}_o^2 = \frac{1}{2} K X_d^2 \Rightarrow X_d = \frac{\dot{X}_o}{\omega_n}$$

$X_d \equiv 1.15 \text{ in} = 0.095 \text{ ft}$

However, there is also a static deflection due to the elevator weight

$$X_S = \frac{W}{K} = \frac{M\,g}{L} \equiv \frac{8000 \text{ lb}}{20000 \text{ lb/in}} = 0.4 \text{ in}$$

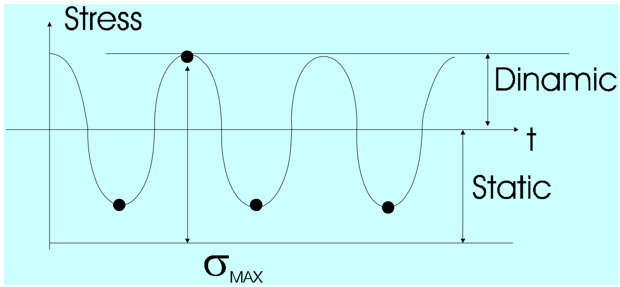


Then the max. amplitude of deflection is

$\boldsymbol{A} = X_S + X_d = 0.4 + 1.15 \text{ in} = 1.558 \text{ in} = 0.1298 \text{ ft}.$

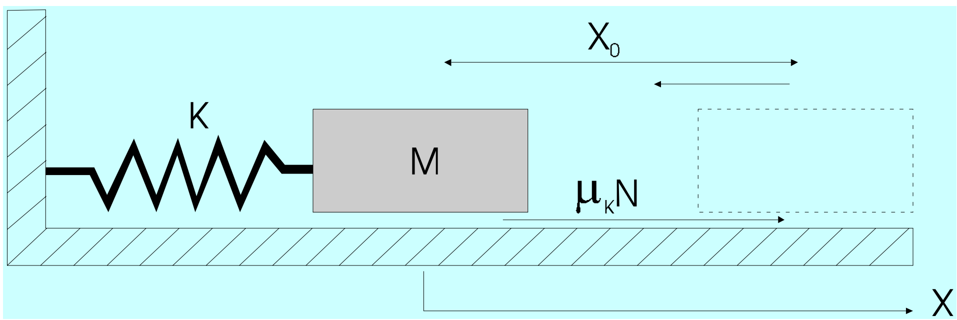
Maximum Stress in cable:

$$\sigma = E\,\varepsilon = E\,\frac{A}{L} = 12 \times 10^6\,\frac{lb}{\text{in}^2} \cdot \frac{1.558\ \text{in}}{50 \times 12\ \text{in}} = 31{,}165\ \text{psi}$$



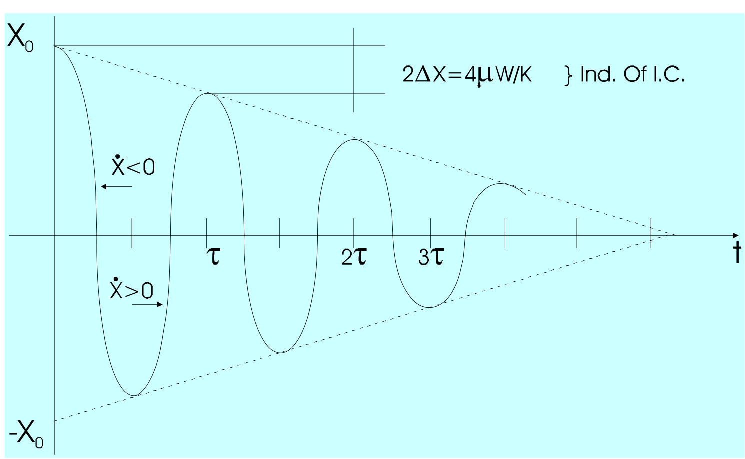
Free Response of a Mass-Spring System with Coulomb Damping (Dry Friction)

Recall that a dry friction force opposes **motion**, and ***F =μN, N = W = Mg***



Consider a mass-spring system resting on top of a surface. The kinetic coefficient of dry friction for relative motion between two surfaces is μ. Assume that at t=0 sec, you release mass M from $X(0)=X_o$. For a not too large friction coefficient, the mass-spring oscillates about the equilibrium position $X=0$ with its natural period $T_n = 2\pi/\omega_n$.

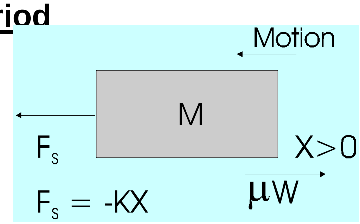
The figure below depicts the motion. The system dynamics is governed by different EOMs if motion is to the left (X decreasing) or to the right (X increasing) since the friction force changes sign. It is of importance to know the amplitude decay (δ) every period of motion and also the time elapsed until the system stops.



Free response of a mass-spring system with dry-friction

Let's analyze the motion for a full period. On the first ½ period, the mass-spring moves to the left and the friction force points towards the right. On the second ½ period, the mass-spring moves to the right and the friction force points towards the left. The amplitude of response shows a finite amplitude decay each ½ period of motion.

MOTION TO THE LEFT: $0 \le t \le \frac{1}{2}\tau = \frac{1}{2} T_n$ **first ½ period**

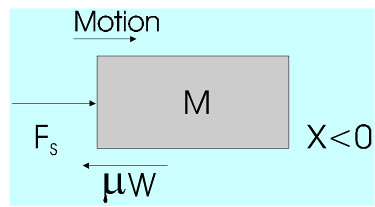


$$M\ddot{X} + KX = F = \mu W \quad (1)$$

$$\text{for} \quad \dot{X} < 0$$

with **I.C.,** $X(0) = X_o, \; \dot{X}(0) = 0$

MOTION TO THE RIGHT: $\frac{1}{2}\tau \le t \le \tau = T_n,$ **second ½ period**



$$M\ddot{X} + KX = -F = -\mu W \quad (2)$$

$$\text{for} \quad \dot{X} > 0$$

with **I.C.**, $X\left(\frac{\tau}{2}\right) = -(X_o - \Delta X), \; \dot{X}\left(\frac{\tau}{2}\right) = 0$

The solution of Eq. (1) or (2) is

$$X(t) = A\cos(\omega_n t) + B\sin(\omega_n t) + F/K$$

MOTION TO THE LEFT: applying the initial conditions obtain

$$X(t) = (X_o - \frac{F}{K})\cos(\omega_n t) + \frac{F}{K} \quad (2)$$

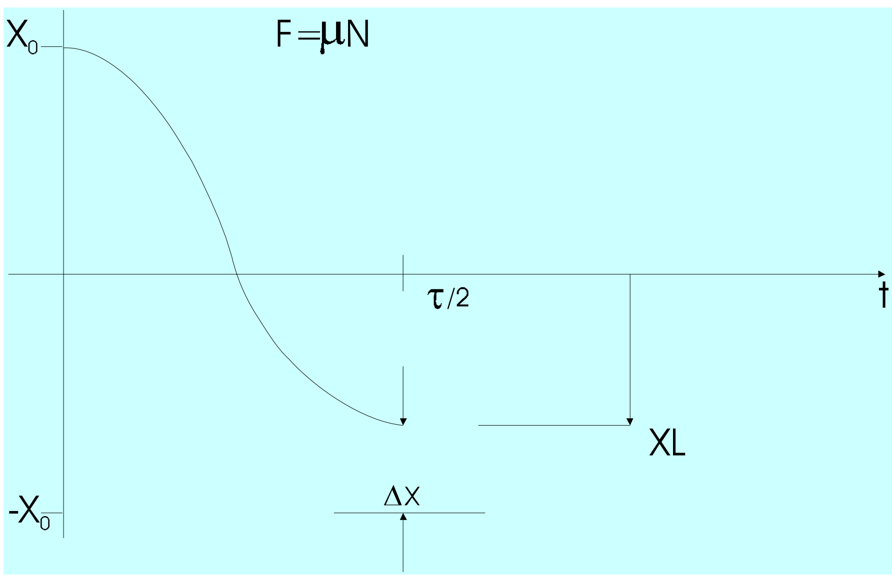
after ½ period, at $t = \frac{\tau}{2} = \frac{\pi}{\omega_n}$, the position of the mass-spring (X_L) is:

$$X(\tau) = X_L = + \quad (X_o - \frac{F}{K}) \ \cos(\pi) \ + \ \frac{F}{K}$$

$$= X_L = -X_o + \frac{2F}{K} = -X_o + \Delta X \qquad (3)$$

Let $\Delta X = \dfrac{2F}{K}$ be the amplitude decay for the first ½ period and note that the velocity $\dot{x}(\tau)=0$

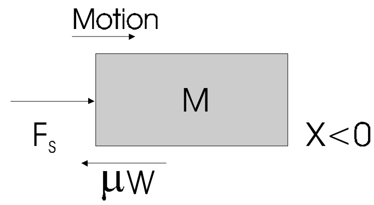
The sketch below shows the response X(t) for the first ½ period of motion



MOTION TO THE RIGHT: $\dot{X} > 0$, second ½ period $\frac{\tau}{2} < t \le \tau$

$$M\ \ddot{X} \ + \ K\ X \ \ = \text{-}\mu\ N \qquad (4)$$

for $\dot{X} > 0$



with **I.C.**, $X\left(\frac{\tau}{2}\right) = -\ (X_o - \Delta X),\ \ \dot{X}\left(\frac{\tau}{2}\right) = 0$

)

or

$$X\left(\frac{\tau}{2}\right) = X_L = X(t'=0); \quad \dot{X}\left(\frac{\tau}{2}\right) = 0 = \dot{X}(t'=0)$$

where for simplicity, define a shift in the time scale as $t' = t - t*$ with

$$t_* = \frac{\pi}{\omega_n} = \frac{\tau}{2} \qquad (5)$$

Then solution of Eq. (4) defining the system dynamics for motion to the right is:

$$X(t') = A' \cos(\omega_n t') + B' \sin(\omega_n t') - F/K \qquad (6)$$

and applying the initial conditions at $t' = 0$, obtain:

$$X(t') = (X_L + \frac{F}{K})\cos(\omega_n t') - \frac{F}{K} \qquad (7)$$

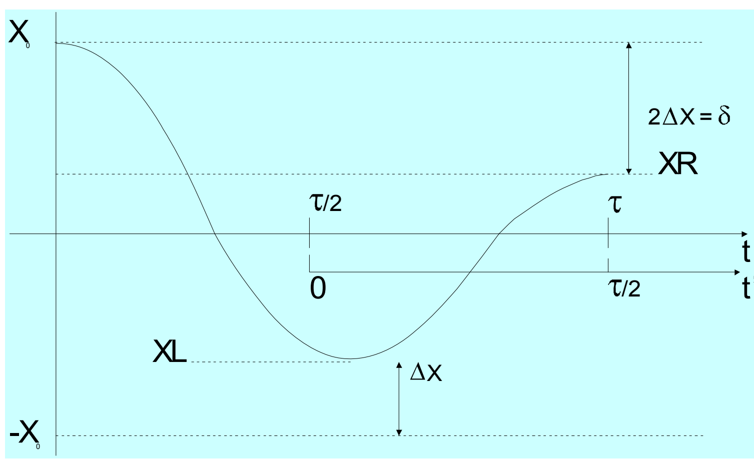
Now, at $t = T_n$ (1 full period), i.e. $t' = ½\,\tau$; i.e. the mass-spring system is at its **rightmost** position (X_R), as shown in the graph below

$$X\left(\frac{\tau}{2}\right) = X_R = (X_L + \frac{F}{K})\cos(\pi) - \frac{F}{K}$$

$$X_R = -X_L - \frac{2F}{K} = -X_L - \Delta X \qquad (8)$$

$$X_R = X_o - 2\Delta X$$

and note that $\dot{X}(\tau) = 0$



Then, after 1 full period of motion the amplitude decays from

X_o to $X_R = X_o - 2\,\Delta X = X_o - \delta$

where $$\delta = 2\Delta X = \frac{4F}{K} = 4\frac{\mu W}{K}$$

The motion proceeds to a stop until

Friction force (μN) > Spring restoring force ($K\,X_{stop}$)

Does the system return to the equilibrium position *X=0* given by the unstretched spring, or is it possible that there maybe more than one equilibrium position?

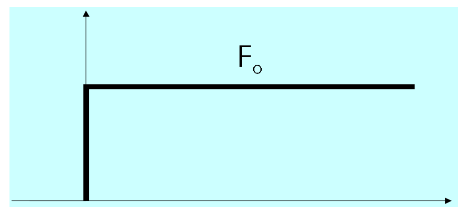
(b) Forced Response of 2nd Order Mechanical System

b.1. Step Force

Let the external force be a suddenly applied step force of magnitude F_o, and consider the system to have initial displacement X_0 and velocity V_0. Then for a system with viscous dissipation mechanism, the equation of motion is

$$M\frac{d^2X}{dt^2}+D\frac{dX}{dt}+KX=F_o \qquad (21)$$

with initial conditions $V(0)=V_o$ and $X(0)=X_o$



Divide Eq. (1) by *M* and define:

$\omega_n=\sqrt{K/M}$: undamped natural frequency of system

$\zeta=\dfrac{D}{D_{cr}}$: viscous damping ratio,

where $D_{cr}=2\sqrt{KM}$ is known as the critical damping value

With these definitions, Eqn. (1) becomes:

$$\frac{d^2X}{dt^2}+2\zeta\,\omega_n\frac{dX}{dt}+\omega_n^2X=\frac{F_o}{M}=\frac{F_o}{K}\omega_n^2 \qquad (22)$$

The solution of the Non-homogeneous Second Order Ordinary Differential Equation with Constant Coefficients is of the form (homogenous + particular):

$$X(t)=X_H+X_P=Ae^{st}+F_o/K \qquad (23)$$

Where A is a constant found from the initial conditions and $X_P=F_o/K$ is the particular solution for the step load.

Note: $X_{ss}=F_o/K$ is equivalent to the static displacement if the force is applied very slowly.

Substitution of (23) into (22) leads to the **CHARACTERISTIC EQUATION** of the system:

$$\left(s^2 + 2\zeta\,\omega_n\, s + \omega_n^2\right) = 0 \tag{25}$$

The roots of this 2nd order polynomial are:

$$s_{1,2} = -\zeta\,\omega_n \mp \omega_n\left(\zeta^2 - 1\right)^{1/2} \tag{26}$$

The nature of the roots (eigenvalues) clearly depends on the value of the damping ratio ζ. Since there are two roots, the solution to the differential equation of motion is now rewritten as:

$$X(t) = A_1\, e^{s_1 t} + A_2\, e^{s_2 t} + F_o / K \tag{27}$$

where A_1, A_2 are constants determined from the initial conditions in displacement and velocity.

From Eq. (27), differentiate three cases:

<u>Underdamped System</u>: $0 < \zeta < 1, \rightarrow D < D_{cr}$

<u>Critically Damped System</u>: $\zeta = 1, \rightarrow D = D_{cr}$

<u>Overdamped System</u>: $\zeta > 1, \rightarrow D > D_{cr}$

Step Forced Response of Underdamped 2nd Order System

For an **underdamped** system, $0 < \zeta < 1$, the roots are complex conjugate (real and imaginary parts), i.e.

$$s_{1,2} = -\zeta\,\omega_n \mp i\,\omega_n\left(1-\zeta^2\right)^{1/2} \tag{28}$$

where $i = \sqrt{-1}$ is the imaginary unit.

The solution for underdamped response of the system adds the homogenous and particular solutions to give:

$$X(t) = e^{-\zeta\,\omega_n t}\left(C_1 \cos\left(\omega_d t\right) + C_2 \sin\left(\omega_d t\right)\right) + X_{ss} \tag{29}$$

where $\omega_d = \omega_n\left(1-\zeta^2\right)^{1/2}$ is the system damped natural frequency.

and $X_{ss} = F_o / K$

At time $t = 0$, the initial conditions are $V(0) = V_o$ and $X(0) = X_o$

Then $$C_1 = \left(X_0 - X_{ss}\right) \text{ and } C_2 = \frac{V_0 + \zeta\,\omega_n C_1}{\omega_d} \tag{30}$$

Note that as $t \to \infty$, $X_{(t)} \to X_{ss} = F_o/K$ for $\zeta > 0$,

i.e. the system response reaches the steady state (static) equilibrium position.

The larger the viscous damping ratio ζ , the fastest the motions will damp out to reach the static position X_{ss}.

Step Forced Response of Undamped 2nd Order System:

For an undamped system, i.e. a conservative system, ζ **= *0***, and the dynamic forced response is given from equation (29) as:

$$X(t) = \left(C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t) \right) + X_{ss} \qquad (31)$$

with $X_{ss} = F_o/K$, $C_1 = (X_0 - X_{ss})$ *and* $C_2 = V_0/\omega_n$ (32)

if the initial displacement and velocity are null, i.e. $\boldsymbol{X_0 = V_0 = 0}$, then

$$X(t) = X_{ss}\left(1 - \cos(\omega_n t)\right) \qquad (33)$$

Note that as $t \to \infty$, $X_{(t)}$ does not approach X_{ss} for $\zeta = 0$.

The system oscillates forever about the static equilibrium position X_{ss} and, the maximum displacement is $2X_{ss}$, i.e. twice the static displacement (F_0/K).

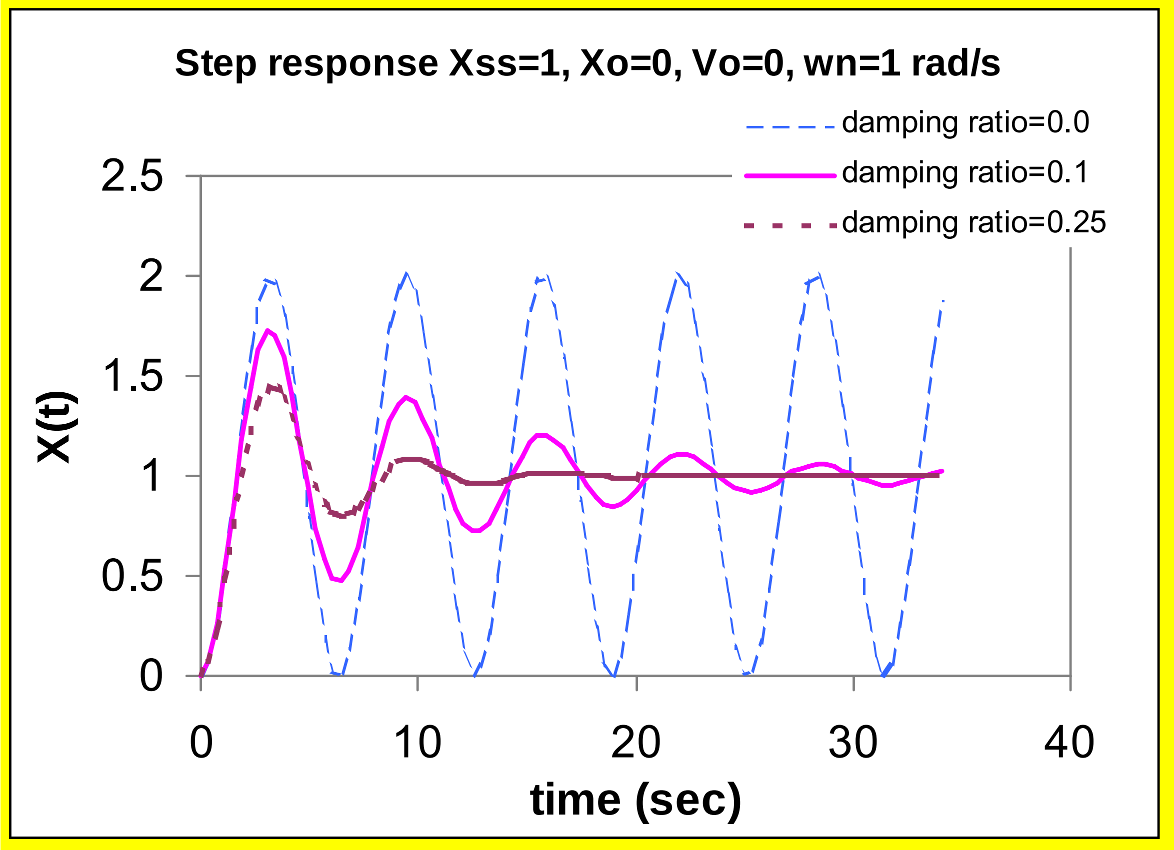
Forced Step Response of Underdamped Second Order System:
damping ratio varies

$X_o = 0,\ \ V_o = 0,\qquad \omega_n = 1.0\ \ rad/s \qquad \zeta = 0,\ 0.1,\ 0.25$

zero initial conditions

$F_o/K = X_{SS} = 1;$

faster response as ζ increases; i.e. as $t \to \infty$, $X \to X_{SS}$ for $\zeta > 0$



Forced Step Response of Overdamped 2nd Order System

For an **overdamped** system, ζ > *1*, the roots of the characteristic eqn. are real and negative, i.e.

$$s_1 = \omega_n \left[-\zeta + \left(\zeta^2 - 1\right)^{1/2} \right];\ s_2 = \omega_n \left[-\zeta - \left(\zeta^2 - 1\right)^{1/2} \right] \qquad (34)$$

The **overdamped forced response** of the system as:

$$X(t) = e^{-\zeta \omega_n t} \left(C_1 \cosh\left(\omega_* t\right) + C_2 \sinh\left(\omega_* t\right) \right) + X_{ss} \qquad (35)$$

where $\omega_* = \omega_n \left(\zeta^2 - 1\right)^{1/2}$. Do not confuse this term with a frequency since motion is NOT oscillatory.

At time *t* = *0*, the initial conditions are $V(0) = V_o$ and $X(0) = X_o$

Then

$$C_1 = \left(X_0 - X_{ss}\right) \text{ and } C_2 = \frac{V_0 + \zeta \omega_n C_1}{\omega_*} \qquad (36)$$

Note that as $t \to \infty$, $X_{(t)} \to X_{ss} = F_o/K$ for ζ > 1, i.e. the steady-state (static) equilibrium position.

An overdamped system does **not** oscillate or vibrate.

The larger the damping ratio ζ , the longer time it takes the system to reach its final equilibrium position X_{ss}.

Forced Step Response of Critically Damped System

For a critically damped system, ζ = *1*, the roots are real negative and identical, i.e.

$$s_1 = s_2 = -\zeta\,\omega_n \tag{37}$$

The step-forced response for critically damped system is

$$X(t) = e^{-\omega_n t}\left(C_1 + t\,C_2\right) + X_{ss} \tag{38}$$

At time *t* = *0*, the initial conditions are $V(0) = V_o$ and $X(0) = X_o$

Then $$C_1 = \left(X_0 - X_{ss}\right) \text{ and } C_2 = V_0 + \omega_n C_1 \tag{39}$$

Note that as $t \to \infty$, $X_{(t)} \to X_{ss} = F_o/K$ for $\zeta > 1$, i.e. the steady-state (static) equilibrium position.

A critically damped system does not oscillate and it is the fastest to reach the steady-state value X_{ss}.

Forced Step Response of Second Order System:

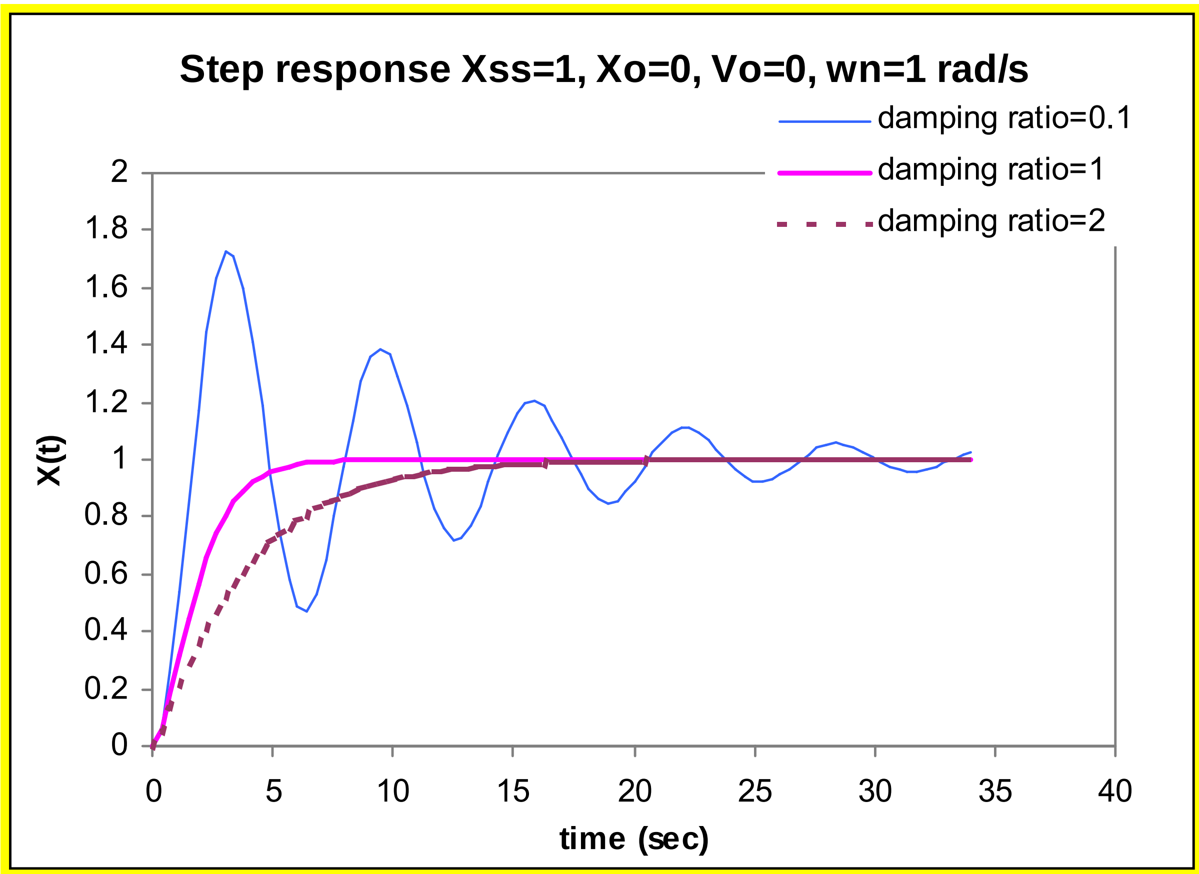
Comparison of Underdamped, Critically Damped and Overdamped system responses

$X_o = 0$, $V_o = 0$, $\omega_n = 1.0$ rad/s $\zeta = 0.1, 1.0, 2.0$

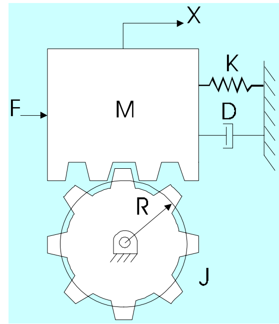
zero initial conditions

$F_o/K = X_{SS} = 1$; (magnitude of s-s response)

Fastest response for $\zeta = 1$. As $t \to \infty$, $X \to X_{SS}$ for $\zeta > 0$



EXAMPLE:



The equation describing the motion and initial conditions for the system shown are:

$$(M + I/R^2)\ddot{X} + D\,\dot{X} + K\,X = F\,, X(0) = \dot{X}(0) = 0$$

Given M=2.0 kg, I=0.01 kg-m^2, D=7.2 N.s/m, K=27.0 N/m, R=0.1 m; and F =5.4 N (a step force),

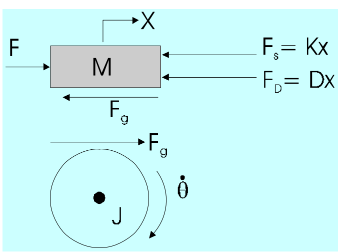
a) Derive the differential equation of motion for the system (as given above).

b) Find the system natural frequency and damping ratio

c) Sketch the dynamic response of the system *X(t)*

d) Find the steady-state value of the response X_{s-s}.

(a) **Using free body diagrams:** Note that θ= X/R is a kinematic constraint. The EOM's are:



$$M\,\ddot{X} = F - K\,X - D\,\dot{X} - F_G \qquad (1)$$

$$I\,\ddot{\theta} = F_G \cdot R \qquad (2)$$

Then from (2) $F_G = I\dfrac{\ddot{\theta}}{R} = I\dfrac{\ddot{X}}{R^2}$ (3) ;

(3) into (1) gives

$$\left(M + \frac{I}{R^2}\right)\ddot{X} + D\dot{X} + K\,X = F \qquad (4)$$

or **Using the Mechanical Energy Method**:

(system kinetic energy): $T = \dfrac{1}{2}\,M\,\dot{X}^2 + \dfrac{1}{2}\,I\,\dot{\theta}^2 = \dfrac{1}{2}\left[M + \dfrac{I}{R^2}\right]\dot{X}^2$ (5)

(system potential energy): $V = \dfrac{1}{2}\,K\,X^2$

(viscous dissipated energy) $E_D = \int D\,\dot{X}^2\;dt$, and External work: $W = \int F\;dX$

Derive identical Eqn. of motion (4) from $\frac{d}{dt}\left(T + V + E_d - W\right) = 0$ (6)

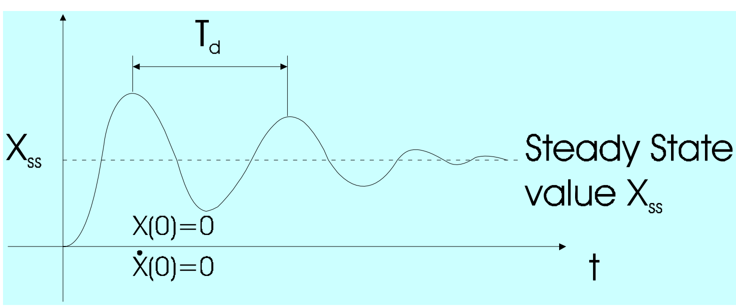
(b) define $M_{eq} = M + \frac{I}{R^2} = 3\;\text{Kg}$, *K* = 27 N/m, *D* = 7.2 N.s/m

and calculate the system natural frequency and viscous damping ratio:

$$\omega_n = \left[\frac{K}{M_{eq}}\right]^{1/2} = 3\;\frac{\text{rad}}{\text{sec}};\;\xi = \frac{D}{2\sqrt{K\;M}} = 0.4,$$ underdamped system

$$\omega_d = \omega_n\;\sqrt{1\text{-}\xi^2} = 2.75\frac{rad}{\text{sec}}$$, and $T_d = \frac{2\pi}{\omega_d} = 2.28\;\text{sec}$ is the damped period of motion

(c) The step response of an underdamped system with I.C.'s $X(0) = \dot{X}(0) = 0$ is:



(d) At steady-state, no motion occurs, $X = X_{SS}$, and $\dot{X} = 0,\;\ddot{X} = 0$

Then

$$X_{ss} = \frac{F}{K} = \frac{5.4\;\text{N}}{27\,\frac{\text{N}}{\text{m}}} \qquad \underline{\underline{X_{ss} = 0.2\;\text{m}}}$$

$$X(t) = X_{ss}\left[1 - e^{-\zeta\,\omega_n\,t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{\zeta^2 - 1}}\sin(\omega_d t)\right)\right]$$